

ON SUBVARIETIES WITH NEF NORMAL BUNDLE

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ABSTRACT. The goal of this work is to study positivity of subvarieties with nef normal bundle in the sense of intersection theory. After Ottem's work on ample subschemes, we introduce the notion of a nef subscheme, which generalizes the notion of a subvariety with nef normal bundle. We show that restriction of a pseudoeffective (resp. big) divisor to a nef subvariety is pseudoeffective (resp. big). We also show that ampleness and nefness are transitive properties.

We define the weakly movable cone as the cone generated by the pushforward of cycle classes of nef subvarieties via proper surjective maps. This cone contains the movable cone and shares similar intersection-theoretic properties with it, thanks to the aforementioned properties of nef subvarieties.

On the other hand, we prove that if $Y \subset X$ is an ample subscheme of codimension r and $D|_Y$ is q -ample, then D is $(q + r)$ -ample. This is analogous to a result proved by Demailly-Peternell-Schneider.

We use the theory of q -ample divisors, as developed by Totaro, throughout the paper.

1. INTRODUCTION

The concept of ampleness of a divisor is central in the subject of algebraic geometry. It plays an important role in intersection theory (Nakai-Moishezon, Kleiman) and various vanishing theorems on cohomologies (Serre, Kodaira, Fujita etc.).

Weakening the Serre vanishing condition, a line bundle \mathcal{L} is defined to be q -ample if given any coherent sheaf \mathcal{F} , there is an m_0 such that

$$H^i(X, \mathcal{F} \otimes \mathcal{L}^{\otimes m}) = 0$$

for $i > q$ and $m > m_0$. Here we assume X is projective over a field of characteristic zero. After the works of Andreotti-Grauert [AG62], Sommese [Som78] and Demailly-Peternell-Schneider [DPS96] on q -ample divisors, Totaro established the basic, yet not elementary properties of q -ample divisors [Tot13]. There is another approach to partial ampleness of a line bundle ([dFKL07], [Kür06]) that we don't pursue it here.

After the extensive work of Hartshorne [Har70], where he studied positivity properties of higher codimension subvarieties, Ottem discovered what is probably the right notion of an ample subscheme [Ott12]. He defined a subscheme of Y of codimension r of a projective scheme to be *ample* if the exceptional divisor in the blowup of X along Y is $(r - 1)$ -ample. It is a natural definition that generalizes many properties of ample divisors [Ott12, Corollary 5.6], which were predicted in Hartshorne's work, while at the same time includes the zero locus of a global section of an ample vector bundle [Ott12, Proposition 4.5].

Our first result sheds more light on the connection between q -ample divisors and ample subschemes:

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Theorem 1.1. *Let X be a projective scheme of dimension n . Let Y be an ample subscheme of X of codimension r . Suppose \mathcal{L} is a line bundle on X , and that its restriction $\mathcal{L}|_Y$ to Y is q -ample, then \mathcal{L} is $(q + r)$ -ample.*

This result can be compared to a result by Demailly-Peternell-Schneider [DPS96, Theorem 3.4]. Given a chain of codimension 1 subvarieties $Y_{n-r} \subset Y_{n-r+1} \subset \cdots \subset Y_{n-1} \subset Y_n = X$, such that for $n - r \leq i \leq n - 1$, there exists an ample divisor Z_i in the normalization of Y_{i+1} , with Y_i being the image of Z_i under the normalization map. They showed if $\mathcal{L}|_{Y_{n-r}}$ is ample, then \mathcal{L} is r -ample, assuming Totaro's results on q -ample divisors.

We now move on to study a weaker positivity condition of a subscheme. Given an lci subvariety $Y \subset X$ with nef normal bundle, we would like to understand its positivity properties in terms of intersection theory. Fulton and Lazarsfeld [FL83] gave an answer to this: They showed that if $\dim Y + \dim Z \geq \dim X$, then $\deg_H(Y \cdot Z) \geq 0$. Here H is an ample divisor.

Now let $Y \subset X$ be an arbitrary subscheme of codimension r and let E be the exceptional divisor in $\text{Bl}_Y X$. We say that Y is *nef* if $(E + \epsilon A)|_E$ is $(r - 1)$ -ample, where A is an ample divisor and $0 < \epsilon \ll 1$. This definition is inspired by Ottem's definition of an ample subscheme [Ott12]. If Y is lci in X , Y is nef if and only if Y has nef normal bundle. We show that

Theorem 1.2. *Let $\iota : Y \hookrightarrow X$ be a nef subvariety of codimension r of a projective variety X . Then the natural map $\iota^* : N^1(X)_{\mathbf{R}} \rightarrow N^1(Y)_{\mathbf{R}}$ induces $\iota^* : \overline{\text{Eff}}^1(X) \rightarrow \overline{\text{Eff}}^1(Y)$ and $\iota^* : \text{Big}(X) \rightarrow \text{Big}(Y)$.*

When Y is a curve with nef normal sheaf, this is a result of Demailly-Peternell-Schneider [DPS96, Theorem 4.1]. We also show that nefness and ampleness are transitive properties without any assumptions on smoothness, thus generalizes Ottem's result [Ott12, Proposition 6.4].

Theorem 1.3. *Let X be a projective scheme of dimension n . If Y is an ample (resp. nef) subscheme of X and Z is an ample (resp. nef) subscheme of Y , then Z is ample (resp. nef) in X .*

In the proof of theorem 1.1 and 1.3, we rely on a generalized version of Fujita vanishing theorem for q -ample divisor that we prove in theorem 3.9. This vanishing theorem is a further generalization of the generalized Fujita-type vanishing theorem that Küronya proved in [Kür13, Theorem C]. However, our methods of proof are completely different. Here we use resolution of the diagonal, as developed by Arapura [Ara06] and Totaro [Tot13].

We then study the cycle classes of nef subvarieties. We use this new notion of nef subvarieties to introduce the notion of the weakly movable cone, $\overline{\text{WMov}}_d(X)$. We define it as the closure of the convex cone that is generated by pushforward of cycle classes of nef subvarieties of dimension d via proper surjective morphisms. We show that the weakly movable cone shares similar properties to that of the movable cone of d -cycles, $\overline{\text{Mov}}_d(X)$.

Theorem 1.4. *Let X be a projective variety of dimension n . For $1 \leq d \leq n - 1$,*

- (1) $\overline{\text{Mov}}_d(X) \subseteq \overline{\text{WMov}}_d(X)$ and $\overline{\text{Mov}}_1(X) = \overline{\text{WMov}}_1(X)$.
- (2) $\overline{\text{Eff}}^1(X) \cdot \overline{\text{WMov}}_d(X) \subseteq \overline{\text{Eff}}_{d-1}(X)$.
- (3) Let H be a big Cartier divisor, $\alpha \in \overline{\text{WMov}}_d(X)$. If $H \cdot \alpha = 0$, then $\alpha = 0$.
- (4) $\text{Nef}^1(X) \cdot \overline{\text{WMov}}_d(X) \subseteq \overline{\text{WMov}}_{d-1}(X)$.

Analogous statements of 2, 3 and 4 hold for the movable cone [FL, Lemma 3.10]. Therefore, it seems sensible to ask whether $\overline{\text{Mov}}_d(X) = \overline{\text{WMov}}_d(X)$. This is true if and only if the cycle

class of any nef subvariety lies in the movable cone. This can be seen as a modified version of the Hartshorne's conjecture A. Hartshorne's conjecture A states that if Y is a smooth subvariety with ample normal bundle of a smooth projective variety X , nY moves in a large algebraic family for n large. This was disproved by Fulton and Lazarsfeld [FL82].

It is unclear what kind of intersection theoretic statements one should expect if we further assume that if Y has ample normal bundle. Voisin gave an example of a subvariety with ample normal bundle such that its cycle class lies on the boundary of the pseudoeffective cone of cycles [Voi10]. On the other hand, Ottem showed that the cycle class of a curve with ample normal bundle lies in the interior of the cone of curves [Ott]. In an upcoming work, we shall study the numerical dimension of a pseudoeffective divisor by restricting it to a subvariety with ample normal bundle.

It is interesting to note that the cone dual to the pseudoeffective cone of d -cycles is not in general pseudoeffective, this is a result by Debarre, Ein, Lazarsfeld and Voisin [DELV11].

All schemes in this work are over a field of characteristic 0.

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2. Q-AMPLE DIVISORS AND AMPLE SUBSCHEMES

In this section, we shall first gather some useful facts about q -ample divisors, then we shall recall Ottem's definition of an ample subscheme and some of its properties.

Let us recall the definition of the definition of a q -ample line bundle.

Definition 2.1 (q -ample line bundle [DPS96],[Tot13]). Let X be a projective scheme. A line bundle \mathcal{L} is q -ample if for any coherent sheaf \mathcal{F} on X , there is an m_0 such that

$$H^i(X, \mathcal{F} \otimes \mathcal{L}^{\otimes m}) = 0$$

for $i > q$ and $m > m_0$.

Lemma 2.2 ([Ott12, Lemma 2.1]). Let X be a projective scheme and fix an ample line bundle $\mathcal{O}(1)$ on X . A line bundle \mathcal{L} is q -ample if and only if for any $l \geq 0$,

$$H^i(X, \mathcal{L}^{\otimes m} \otimes \mathcal{O}(-l)) = 0$$

for $m \gg 0$.

We shall start with the definition of a Koszul-ample line bundle. The details are not very important in this paper, but they are included for the sake of completeness. One useful fact is that any large tensor power of an ample line bundle is $2n$ -Koszul-ample, where n is the dimension of the underlying projective scheme [Bac86].

Definition 2.3 (Koszul-ameness [Tot13, Section 1]). Let X be a projective scheme of dimension n , and that the ring of regular function $\mathcal{O}(X)$ on X is a field (e.g. X is connected and reduced). Given a very ample line bundle $\mathcal{O}_X(1)$, we say that it is N -Koszul ample if the homogeneous coordinate ring $A = \bigoplus_j H^0(X, \mathcal{O}_X(j))$ is N -Koszul, i.e. there is a resolution

$$\cdots \rightarrow M_1 \rightarrow M_0 \rightarrow k \rightarrow 0$$

where M_i is a free A -module, generated in degree i , where $i \leq N$.

Definition 2.4 (q -T-ampleness [Tot13, Definition 6.1]). Let X be a projective scheme of dimension n . Suppose the ring of regular functions of X , $\mathcal{O}(X)$ is a field. We fix a $2n$ -Koszul-ample line bundle $\mathcal{O}_X(1)$ on X . We say that a line bundle \mathcal{L} is q -T-ample if there is a positive integer N , such that

$$H^{q+i}(X, \mathcal{L}^{\otimes N}(-n-i)) = 0,$$

for $0 \leq i \leq n-q$.

Totaro showed that q -T-ampleness is the same as q -ampleness [Tot13, Theorem 6.3]. Even though the q -T-ampleness notion may appear technical, the equivalence is the key result of his paper. It reduces the problem of showing a line bundle being q -ample to checking the vanishing of finitely many cohomology groups. Using the notion of q -T-ampleness, Totaro showed that q -ampleness is Zariski open [Tot13, Theorem 8.1]. We can extend the definition to \mathbf{R} -Cartier divisors.

Definition 2.5 (q -ample \mathbf{R} -divisors). Let X be a projective scheme. An \mathbf{R} -Cartier divisor on X is q -ample if D is numerically equivalent to $cL + A$ with L a q -ample line bundle, $c \in \mathbf{R}_{>0}$, A an ample \mathbf{R} -Cartier divisor.

Based on the work of Demailly, Peternell and Schneider, Totaro also proved that

Theorem 2.6 ([Tot13, Theorem 8.3]). *An integral divisor is q -ample if and only if its associated line bundle is q -ample. The q -ample \mathbf{R} -divisors in $N^1(X)_{\mathbf{R}}$ defines an open cone (but not convex in general) and that the sum of a q -ample \mathbf{R} -divisor and a r -ample \mathbf{R} -divisor is $(q+r)$ -ample.*

These facts are non-trivial. We shall use the notion of q -T-ampleness to prove proposition 2.8.

We note that $(n-1)$ -ampleness admits a pleasant geometric interpretation, which we shall use a few times in this paper.

Theorem 2.7 ([Tot13, Theorem 9.1]). *Let X be a projective variety of dimension n . A line bundle \mathcal{L} on X is $(n-1)$ -ample if and only if $[\mathcal{L}^{\vee}] \in N^1(X)$ does not lie in the pseudoeffective cone.*

We need the following result on the positivity of the pullback of a q -ample divisor.

Proposition 2.8 (Pullback of a q -ample divisor). *Let $f : X' \rightarrow X$ be a morphism of projective schemes. Let D be a q -ample divisor on X , and let A be a relatively (to f) ample divisor on X' . Then $mf^*D + A$ is q -ample, for $m \gg 0$.*

Proof. First, let us show that it suffices to prove the proposition in the case when both X and X' are irreducible and reduced. Note that a line bundle is q -ample on X' if and only if it is q -ample when restricting to each irreducible component of X' [Ott12, Proposition 2.3.i,ii]. We can now assume X' is integral. Let X_1 be an irreducible component of X that contains the image of X' . The map $X' \rightarrow X$ factors through X_1 , and $D|_{X_1}$ is again q -ample.

Now we can assume both X and X' are integral. In fact, we shall prove that $mf^*D + A$ is q -T-ample, for $m \gg 0$. In other words, we shall show that for $m \gg 0$, there is a positive integer r , such that

$$H^{q+a}(X', \mathcal{O}_{X'}(r(mf^*D + A)) \otimes \mathcal{O}_{X'}(-n-a)) = 0$$

for $1 \leq a \leq n-q$. Here $\mathcal{O}_{X'}(1)$ is a $2n$ -Koszul-ample line bundle on X' , where $n = \dim X'$.

Using the relative ampleness of A , one can find an integer r such that

$$R^j f_*(\mathcal{O}_{X'}(rA) \otimes \mathcal{O}_{X'}(-n-a)) = 0,$$

for $j > 0$ and $1 \leq a \leq n - q$. The Leray spectral sequence then says

$$(2.1) \quad H^{q+a}(X', \mathcal{O}_{X'}(r(mf^*D + A)) \otimes \mathcal{O}_{X'}(-n-a)) \\ \cong H^{q+a}(X, \mathcal{O}_X(rmD) \otimes f_*(\mathcal{O}_{X'}(rA) \otimes \mathcal{O}_{X'}(-n-a))).$$

The right hand side group vanishes for all big m , by the q -ampleness of rD . \square

We now review the definition of ample subscheme, given by Ottem:

Definition 2.9 (Ample subscheme [Ott12, Definition 3.1]). Let X be a projective scheme. Let Y be a closed subscheme of X of codimension r and let $\pi : \text{Bly } X \rightarrow X$ be the blowup of X with center Y . We say that Y is an *ample subscheme of X* if the exceptional divisor E of π is $(r-1)$ -ample in $\text{Bly } X$.

We shall follow his definition in this paper. An example of an ample subscheme would be the zero locus (of codimension r) of a section of an ample vector bundle of rank r [Ott12, Proposition 4.5]. On the other hand, many good properties listed in Hartshorne's book [Har70, p.XI] are satisfied under this definition. Before stating some of these properties, we need the definition of *cohomological dimension* of a scheme U : it refers to the number

$$\text{cd}(U) := \max\{i \in \mathbb{Z}_{\geq 0} \mid H^i(U, \mathcal{F}) \neq 0, \text{ for some coherent sheaf } \mathcal{F}\}$$

Theorem 2.10. *Let Y be a smooth closed subscheme of a smooth projective scheme X .*

- (1) *Y is ample if and only if its normal bundle is ample and the cohomological dimension of the complement is $r-1$.*

Assume further that Y is an ample subscheme in X . Then

- (2) *Generalized Lefschetz hyperplane theorem with rational coefficient holds, i.e. $H^i(X, \mathbb{Q}) \rightarrow H^i(Y, \mathbb{Q})$ is an isomorphism for $i < \dim Y$ and is an injection for $i = \dim Y$.*
- (3) *Y is numerically positive, i.e. $Y \cdot Z > 0$ for any effective cycle Z of dimension r .*
- (4) *$H^i(X, \mathcal{F}) \rightarrow H^i(\hat{X}, \hat{\mathcal{F}})$ is an isomorphism for $i < \dim Y$ and is injective for $i = \dim Y$. Here \hat{X} is the formal completion of X along Y , \mathcal{F} is a locally free sheaf on X and $\hat{\mathcal{F}}$ is its restriction to \hat{X} .*

Proof. [Ott12, Theorem 5.4], [Ott12, Corollary 5.3] and [Har70, Chapter III, Theorem 3.4] give 1, 2 and 4 respectively. For 3, since the cohomological dimension of $(X - Y) = r-1$, Y meets any effective cycle of dimension r . We can then apply the result of Fulton and Lazarsfeld [Laz04, Corollary 8.4.3], which says if Y has ample normal bundle and Y meets Z , where Z is for an effective cycle of complementary dimension to that of Y , then $Y \cdot Z > 0$. \square

The above list of properties is incomplete, for a more complete picture, c.f. [Ott12].

3. PARTIAL REGULARITY AND A FUJITA-TYPE VANISHING THEOREM FOR q -AMPLE DIVISORS

In this section, we shall quickly go through the results in section 2 and 3 in Totaro's paper [Tot13]. There Totaro developed on Arapura's idea [Ara06] on using resolution of the diagonal to study Castelnuovo-Mumford regularity of a sheaf. Using these ideas, we shall provide a weak extension of a vanishing theorem for q -ample line bundles proved by Totaro [Tot13, Theorem 6.4] (theorem 3.7). From this, we prove a generalization of the

Fujita vanishing theorem (theorem 3.9) to the q -ample divisors setting. It also generalizes the Fujita-type vanishing theorem that Küronya proved [Kür13, Theorem C]. We shall later apply this theorem to prove theorem 5.1, as well as theorems 4.10 and 4.12.

In this section, we assume X to be a projective scheme of dimension n over a field, with the ring of regular functions on X being a field. Furthermore, we fix a $2n$ -Koszul-ample line bundle $\mathcal{O}_X(1)$ on X .

Theorem 3.1 (Totaro [Tot13, Theorem 2.1]). *On $X \times_k X$, we have the following exact sequence of coherent sheaves:*

$$(3.1) \quad \mathcal{R}_{2n-1} \boxtimes \mathcal{O}_X(-2n+1) \rightarrow \cdots \rightarrow \mathcal{R}_1 \boxtimes \mathcal{O}_X(-1) \rightarrow \mathcal{R}_0 \boxtimes \mathcal{O}_X \rightarrow \mathcal{O}_\Delta \rightarrow 0,$$

where $\Delta \subset X \times_k X$ is the diagonal. Here all the \mathcal{R}_i 's are locally free sheaves on X that can be fit into short exact sequences:

$$(3.2) \quad 0 \rightarrow \mathcal{R}_{i+1} \otimes \mathcal{O}_X(-1) \rightarrow B_{i+1} \otimes_k \mathcal{O}_X(-1) \rightarrow \mathcal{R}_i \rightarrow 0,$$

where the B_{i+1} 's are k -vector spaces.

Lemma 3.2. [Tot13, Lemma 3.1] *Let \mathcal{E} and \mathcal{F} be a locally free sheaf and a coherent sheaf on X respectively. Suppose that for each pair of integers $0 \leq a \leq 2n-i$ and $b \geq 0$, either $H^b(\mathcal{E} \otimes \mathcal{R}_a) = 0$ or $H^{i+a-b}(\mathcal{F}(-a)) = 0$. Then $H^i(\mathcal{E} \otimes \mathcal{F}) = 0$.*

Sketch of proof. After tensoring with $\mathcal{E} \boxtimes \mathcal{F}$, the sequence (3.1) remains exact, we now apply Künneth's formula. \square

Definition 3.3 (Partial regularity [Kee03, Definition 2.1]). We fix a $2n$ -Koszul ample line bundle $\mathcal{O}_X(1)$. Let \mathcal{G} be a coherent sheaf on X and let q be any integer greater than or equal to 0. We say that \mathcal{G} is q -regular if the following holds:

$$(3.3) \quad H^{q+i}(X, \mathcal{G} \otimes \mathcal{O}_X(-i)) = 0$$

for all $1 \leq i \leq n - q$.

We set

$$\text{reg}^q(\mathcal{F}) = \inf\{m \in \mathbb{Z} \mid \mathcal{F} \otimes \mathcal{O}_X(m) \text{ is } q\text{-regular.}\}$$

When $q = 0$, this is just the usual Castelnuovo-Mumford regularity of the sheaf \mathcal{F} , relative to $\mathcal{O}_X(1)$. It is clear that $\text{reg}^q(\mathcal{F}) \in [-\infty, +\infty)$, by the ampleness of $\mathcal{O}_X(1)$.

Lemma 3.4 ([Kee03, Lemma 2.2]). *If \mathcal{F} is q -regular, then $\mathcal{F} \otimes \mathcal{O}_X(1)$ is also q -regular.*

Lemma 3.5 ([Tot13, Lemma 3.3]). *If \mathcal{F} is a q -regular coherent sheaf on X , then*

$$H^j(X, \mathcal{F} \otimes \mathcal{R}_i) = 0,$$

for $j > q$ and $i < n + j$. Here, we are referring to the \mathcal{R}_i 's that appear in lemma 3.2.

We next generalize [Tot13, Theorem 3.4].

Theorem 3.6 (Subadditivity of partial regularity). *Let \mathcal{E} and \mathcal{F} be a locally free sheaf and a coherent sheaf on X respectively, then*

$$\text{reg}^q(\mathcal{E} \otimes \mathcal{F}) \leq \text{reg}^l(\mathcal{E}) + \text{reg}^{q-l}(\mathcal{F})$$

for any $0 \leq l \leq q$.

Proof. Replacing \mathcal{E} and \mathcal{F} by $\mathcal{E} \otimes \mathcal{O}_X(k)$ and $\mathcal{F} \otimes \mathcal{O}_X(k')$ respectively, where k and k' are sufficiently large, we may assume \mathcal{E} and \mathcal{F} are l - and $(q-l)$ -regular, respectively. We want to show

$$H^{q+i}(X, \mathcal{E} \otimes \mathcal{F} \otimes \mathcal{O}_X(-i)) = 0,$$

for $1 \leq i \leq n-q$. We now apply lemma 3.2.

Case 1. $b > l$ and $a < n+b$.

By lemma 3.5,

$$H^b(X, \mathcal{E} \otimes \mathcal{R}_a) = 0.$$

Case 2. $b > l$ and $n+b \leq a \leq 2n-(q+i)$.

Since $q+i+a-b \geq q+i+n > n$,

$$H^{(q+i)+a-b}(X, \mathcal{F} \otimes \mathcal{O}_X(-a-i)) = 0,$$

for dimensional reason.

Case 3. $0 \leq b \leq l$ and $0 \leq a \leq 2n-(q+i)$.

We have $q-b \geq q-l$, and

$$H^{(q-b)+a+i}(X, \mathcal{F} \otimes \mathcal{O}_X(-a-i)) = 0,$$

by $(q-l)$ -regularity of \mathcal{F} and lemma 3.4. □

We next prove an analogue of [Tot13, Theorem 6.4]. This will play a crucial role in proving theorem 5.1.

Theorem 3.7 (Uniform vanishing). *Let \mathcal{L} be a q -ample line bundle on X . Then for any N , there is an integer m_N , such that, for any coherent sheaf \mathcal{F} on X with $\text{reg}^{q'}(\mathcal{F}) \leq N$,*

$$H^i(X, \mathcal{F} \otimes \mathcal{L}^{\otimes m}) = 0$$

for $i > q+q'$ and $m > m_N$.

Proof. Fix an integer i such that $q+q' < i \leq n$, by lemma 3.2, it is enough to show that there is an M , depending only on the choice of N , but not the coherent sheaf \mathcal{F} , such that for $m > M$, $0 \leq a \leq 2n-i$ and $b \geq 0$, either $H^b(X, \mathcal{L}^{\otimes m} \otimes \mathcal{O}_X(-N) \otimes \mathcal{R}_a) = 0$ or $H^{i+a-b}(X, \mathcal{F} \otimes \mathcal{O}_X(N-a)) = 0$. Here \mathcal{F} is any coherent sheaf with $\text{reg}^{q'}(\mathcal{F}) \leq N$.

Case 1. $b > q$ and $0 \leq a < n+b$.

Using the q -ampleness of \mathcal{L} , there is an m_N , such that we have

$$H^{q+j}(X, \mathcal{L}^{\otimes m} \otimes \mathcal{O}_X(-N-j)) = 0$$

for all $1 \leq j \leq n-q$ and $m > m_N$, i.e. $\mathcal{L}^{\otimes m} \otimes \mathcal{O}_X(-N)$ is q -regular for all $m > m_N$. Now lemma 3.5 says

$$H^b(X, \mathcal{L}^{\otimes m} \otimes \mathcal{O}_X(-N) \otimes \mathcal{R}_a) = 0$$

for all $m > m_N$, $b > q$ and $a < n+b$.

Case 2. $b > q$ and $n+b \leq a \leq 2n-i$.

We have $i+a-b \geq i+n > n$, and

$$H^{i+a-b}(X, \mathcal{F} \otimes \mathcal{O}_X(N-a)) = 0$$

for dimensional reason.

Case 3. $0 \leq b \leq q$ and $0 \leq a \leq 2n - i$.

We have $i - b > q'$, and $H^{(i-b)+a}(X, \mathcal{F} \otimes \mathcal{O}_X(N - a)) = 0$ by the partial regularity assumption of \mathcal{F} and lemma 3.4. This proves the theorem. \square

Lemma 3.8. *There is an N such that $\text{reg}^0(\mathcal{P}) \leq N$ for any nef line bundle \mathcal{P} on X .*

Proof. By the Fujita vanishing theorem, there is an N such that

$$H^a(X, \mathcal{O}_X(N - a) \otimes \mathcal{P}) = 0$$

for $a > 0$ and any nef line bundle \mathcal{P} . \square

We prove a Fujita-type vanishing theorem for q -ample divisors. It is a generalization of the Fujita-type vanishing theorem that Küronya proved in [Kür13, Theorem C], thanks to the fact that a divisor D is q -ample if and only if its restriction to its augmented base locus $D|_{\mathbf{B}_+(D)}$ is q -ample [Bro12]. Note that we do not assume $\mathcal{O}(Z)$ is a field in the following.

Theorem 3.9 (Fujita-type vanishing theorem for q -ample divisors). *Let Z be a projective scheme of dimension n . Let \mathcal{L}_i be q_i -ample line bundles on Z , $1 \leq i \leq k$ and let \mathcal{F} be a coherent sheaf on Z . Then for any $(k-1)$ -tuple $(M_2, \dots, M_k) \in \mathbb{Z}^{k-1}$, there is an M_1 , such that*

$$H^i(Z, \mathcal{F} \otimes \mathcal{L}_1^{\otimes m_1} \otimes \mathcal{L}_2^{\otimes m_2} \otimes \dots \otimes \mathcal{L}_k^{\otimes m_k} \otimes \mathcal{P}) = 0$$

for $i > \sum_{i=1}^k q_i$, $m_i \geq M_i$, where $1 \leq i \leq k$, and any nef line bundle \mathcal{P} on Z .

Proof. We can assume that Z is connected. It suffices to prove the lemma assuming that Z is also reduced. Indeed, let \mathcal{N} be the nilradical ideal sheaf of Z , and chase through the following exact sequence:

$$\begin{aligned} 0 \rightarrow \mathcal{N}^{e+1} \cdot \mathcal{F} \otimes \mathcal{L}_1^{\otimes m_1} \otimes \mathcal{L}_2^{\otimes m_2} \otimes \dots \otimes \mathcal{L}_k^{\otimes m_k} \otimes \mathcal{P} \\ \rightarrow \mathcal{N}^e \cdot \mathcal{F} \otimes \mathcal{L}_1^{\otimes m_1} \otimes \mathcal{L}_2^{\otimes m_2} \otimes \dots \otimes \mathcal{L}_k^{\otimes m_k} \otimes \mathcal{P} \\ \rightarrow (\mathcal{N}^e \cdot \mathcal{F} / \mathcal{N}^{e+1} \cdot \mathcal{F}) \otimes \mathcal{L}_1^{\otimes m_1} \otimes \mathcal{L}_2^{\otimes m_2} \otimes \dots \otimes \mathcal{L}_k^{\otimes m_k} \otimes \mathcal{P} \rightarrow 0. \end{aligned}$$

Note that $(\mathcal{N}^e \cdot \mathcal{F} / \mathcal{N}^{e+1} \cdot \mathcal{F}) \otimes \mathcal{L}_1^{\otimes m_1} \otimes \mathcal{L}_2^{\otimes m_2} \otimes \dots \otimes \mathcal{L}_k^{\otimes m_k} \otimes \mathcal{P}$ is a coherent sheaf on Z_{red} , and that $\mathcal{N}^e = 0$ for $e \gg 0$.

Since \mathcal{L}_i is q_i -ample,

$$H^{q_i+a}(Z, \mathcal{L}_i^{\otimes m_i} \otimes \mathcal{O}(-a)) = 0$$

for $m_i \gg 0$ and $1 \leq a \leq n - q_i$. This says $\text{reg}^{q_i}(\mathcal{L}_i^{\otimes m_i}) \leq 0$ for all $m_i \gg 0$. Therefore, there is an N_i such that $\text{reg}^{q_i}(\mathcal{L}_i^{\otimes m_i}) \leq N_i$ for all $m_i \geq M_i$. We apply theorem 3.6 and lemma 3.8 to see that $\text{reg}^{\sum_{i=2}^k q_i}(\mathcal{F} \otimes \mathcal{L}_2^{\otimes m_2} \otimes \dots \otimes \mathcal{L}_k^{\otimes m_k} \otimes \mathcal{P}) \leq \text{reg}(\mathcal{F}) + \sum_{i=2}^k N_i + N$ for all $m_i \geq M_i$, where $2 \leq i \leq k$ and N is the one mentioned in lemma 3.8. Now, we may apply theorem 3.7 to get the desired result. \square

Suppose we are only interested in the vanishing of the top cohomology group only, we may relax the assumption in theorem 3.9 a bit. We shall use this to prove theorem 6.1.

Proposition 3.10. *Let Z be a projective scheme of dimension n . Let \mathcal{L}_1 and \mathcal{L}_i be line bundles on Z that are q_1 -ample and q_i -almost ample respectively, where $2 \leq i \leq k$ and $\sum_{i=1}^k q_i \leq n-1$. Then for any coherent sheaf \mathcal{F} on Z and any $(k-1)$ -tuple $(M_i)_{2 \leq i \leq k} \in \mathbb{Z}^{k-1}$, there is an M_1 such that,*

$$H^n(Z, \mathcal{F} \otimes \mathcal{L}_1^{\otimes m_1} \otimes \bigotimes_{i=2}^k \mathcal{L}_i^{\otimes m_i}) = 0$$

for $m_i \geq M_i$.

Proof. Let us first reduce to the case where Z is integral. Indeed, argue as in the proof of theorem 3.9, we may assume Z is reduced. Suppose $Z = \bigcup_{i=1}^k Z_i$, where Z_i are the irreducible components of Z . Let \mathcal{I} be the ideal sheaf of $Z_1 \subset Z$. Consider the short exact sequence

$$0 \rightarrow \mathcal{I} \cdot \mathcal{F} \rightarrow \mathcal{F} \rightarrow \mathcal{F}/\mathcal{I} \cdot \mathcal{F} \rightarrow 0.$$

Note that $\mathcal{I} \cdot \mathcal{F}$ and $\mathcal{F}/\mathcal{I} \cdot \mathcal{F}$ are supported on $\bigcup_{i=2}^k Z_i$ and Z_1 respectively. We then tensor the above short exact sequence with $\mathcal{L}_1^{\otimes m_1} \otimes \bigotimes_{i=2}^k \mathcal{L}_i^{\otimes m_i}$ and induct on the number of irreducible components of Z . Therefore, we may assume that Z is irreducible as well.

Now we assume Z is a projective variety. We can find a surjection $\oplus \mathcal{O}_Z(a) \rightarrow \mathcal{F}$, where $\mathcal{O}_Z(1)$ is an ample line bundle on Z . Thus it suffices to prove the case when \mathcal{F} is a line bundle \mathcal{M} . Let ω_Z be the dualizing sheaf of Z [Har77, III.7]. We have

$$H^n(Z, \mathcal{M} \otimes \mathcal{L}_1^{\otimes m_1} \otimes \bigotimes_{i=2}^k \mathcal{L}_i^{\otimes m_i}) \cong H^0(Z, \mathcal{M}^\vee \otimes \mathcal{L}_1^{\otimes -m_1} \otimes \bigotimes_{i=2}^k \mathcal{L}_i^{\otimes -m_i} \otimes \omega_Z)^\vee$$

We can embed $\omega_Z \hookrightarrow \mathcal{O}(j)$ [Tot13, Proof of Theorem 9.1]. This reduces to proving the vanishing of $H^0(Z, \mathcal{M}^\vee \otimes \mathcal{O}(j) \otimes \mathcal{L}_1^{\otimes -m_1} \otimes \bigotimes_{i=2}^k \mathcal{L}_i^{\otimes -m_i})$. We may find an M_1 such that $\mathcal{L}_1^{\otimes m_1} \otimes \bigotimes_{i=2}^k \mathcal{L}_i^{\otimes M_i} \otimes \mathcal{M} \otimes \mathcal{O}(-j)$ is q_1 -ample for $m_1 \geq M_1$, by theorem 2.6. By theorem 2.6 again, $\bigotimes_{i=2}^k \mathcal{L}_i^{\otimes m_i} \otimes \mathcal{L}_1^{\otimes m_1} \otimes \mathcal{M} \otimes \mathcal{O}(-j)$ is $(n-1)$ -ample for $m_i \geq M_i$ and $m_1 \geq M_1$. By theorem 2.7, $\bigotimes_{i=2}^k \mathcal{L}_i^{\otimes -m_i} \otimes \mathcal{L}_1^{\otimes -m_1} \otimes \mathcal{M}^\vee \otimes \mathcal{O}(j)$ is not pseudoeffective for $m_i \geq M_i$. Therefore, it cannot have any global sections. \square

4. NEF SUBSCHEMES

In this section, we shall define the notion of nef subschemes. We shall show that ampleness and nefness are transitive properties: If Z is an ample (resp. nef) subscheme of Y and Y is an ample (resp. nef) subscheme of X , then Z is an ample (resp. nef) subscheme of X (theorems 4.10 and 4.12). We shall study them more closely in later sections. To streamline the arguments, we first make the following definition, which generalizes the notion of a nef divisor.

Definition 4.1 (q -almost ample). Let X be a projective scheme, D an \mathbf{R} -Cartier divisor on X , A an ample Cartier divisor on X . We say that D is q -almost ample if $D + \epsilon A$ is q -ample for $0 < \epsilon \ll 1$.

The definition is clearly independent of the choice of A and D is 0-almost ample if and only if D is nef.

Ottem observed that ampleness of a vector bundle \mathcal{E} can be expressed in terms of q -ampleness of $\mathbb{P}(\mathcal{E}^\vee)$ [Ott12, Proposition 4.1]. We give the straightforward generalization to the case when the vector bundle is nef.

Proposition 4.2. *Let \mathcal{E} be a vector bundle of rank r on a projective scheme X . Then \mathcal{E} is ample (resp. nef) if and only if $\mathcal{O}_{\mathbb{P}(\mathcal{E}^\vee)}(-1)$ is $(r-1)$ -ample. (resp. $(r-1)$ -almost ample.)*

Proof. Let $\pi' : \mathbb{P}(\mathcal{E}^\vee) \rightarrow X$ and $\pi : \mathbb{P}(\mathcal{E}) \rightarrow X$ be the natural projection maps. Using [Har77, Exercise III.8.4], we have for $m > 0$,

$$R^j \pi'_* \mathcal{O}_{\mathbb{P}(\mathcal{E}^\vee)}(-m-r) \cong \begin{cases} \text{Sym}^m \mathcal{E} \otimes \det(\mathcal{E}) & \text{for } j = r-1 \\ 0 & \text{otherwise.} \end{cases}$$

Here we implicitly used the isomorphism $(\mathrm{Sym}^m \mathcal{E}^\vee)^\vee \cong \mathrm{Sym}^m \mathcal{E}$ which holds when the ground field is of characteristic 0.

Therefore we have the isomorphisms

$$(4.1) \quad H^{r-1+i}(\mathbb{P}(\mathcal{E}^\vee), \mathcal{O}_{\mathbb{P}(\mathcal{E}^\vee)}(-m-r) \otimes \pi'^*(\mathcal{F} \otimes \det \mathcal{E}^\vee)) \cong H^i(X, \mathrm{Sym}^m \mathcal{E} \otimes \mathcal{F}) \\ \cong H^i(\mathbb{P}(\mathcal{E}), \mathcal{O}_{\mathbb{P}(\mathcal{E})}(m) \otimes \pi^* \mathcal{F}),$$

where \mathcal{F} is locally free on X , $i > 0$ and $m > 0$.

If $\mathcal{O}_{\mathbb{P}(\mathcal{E}^\vee)}(-1)$ is $(r-1)$ -ample, then the above observation shows that $\mathcal{O}_{\mathbb{P}(\mathcal{E})}(1)$ is ample. Indeed, any line bundle on $\mathbb{P}(\mathcal{E})$ can be expressed as $\pi^* \mathcal{L} \otimes \mathcal{O}_{\mathbb{P}(\mathcal{E})}(l)$.

Suppose $\mathcal{O}_{\mathbb{P}(\mathcal{E}^\vee)}(-1)$ is $(r-1)$ -almost ample. Choose an ample divisor A on X , to show that $\mathcal{O}_{\mathbb{P}(\mathcal{E})}(1)$ is nef, we want to check that $\mathcal{O}_{\mathbb{P}(\mathcal{E})}(k) \otimes \pi^* \mathcal{O}(A)$ is ample for all $k > 0$. By replacing A with a large multiple, we may assume $\mathcal{O}_{\mathbb{P}(\mathcal{E})}(1) \otimes \pi^* \mathcal{O}(A)$ is ample. We apply lemma 2.2 and fix an $l \geq 0$. Observe that we have the following isomorphisms given by (4.1):

$$H^i(\mathbb{P}(\mathcal{E}), \mathcal{O}_{\mathbb{P}(\mathcal{E})}(mk-l) \otimes \pi^* \mathcal{O}((m-l)A)) \\ \cong H^{r-1+i}(\mathbb{P}(\mathcal{E}^\vee), \mathcal{O}_{\mathbb{P}(\mathcal{E}^\vee)}(-mk-r+l) \otimes \pi'^*(\mathcal{O}((m-l)A) \otimes \det \mathcal{E}^\vee)),$$

where $i, m > 0$. The latter term vanishes for $m \gg 0$ since $\mathcal{O}_{\mathbb{P}(\mathcal{E}^\vee)}(-k) \otimes \pi'^* \mathcal{O}(A)$ is $(r-1)$ -ample for any $k > 0$. This shows that $\mathcal{O}_{\mathbb{P}(\mathcal{E})}(1)$ is nef.

Similarly, we may also assume $\mathcal{O}_{\mathbb{P}(\mathcal{E}^\vee)}(1) \otimes \pi'^* \mathcal{O}(A)$ is ample.

If \mathcal{E} is ample, we fix an $l \geq 0$, we have the following isomorphisms of cohomology groups,

$$H^{r-1+i}(\mathbb{P}(\mathcal{E}^\vee), \mathcal{O}_{\mathbb{P}(\mathcal{E}^\vee)}(-m-l) \otimes \pi'^* \mathcal{O}(-lA)) \\ \cong H^i(\mathbb{P}(\mathcal{E}), \mathcal{O}_{\mathbb{P}(\mathcal{E})}(m+l-r) \otimes \pi^*(\mathcal{O}(-lA) \otimes \det \mathcal{E})),$$

the latter term vanishes for $i > 0$ and $m \gg 0$, which says $\mathcal{O}_{\mathbb{P}(\mathcal{E}^\vee)}(-1)$ is $(r-1)$ -ample.

If \mathcal{E} is nef, we fix $l \geq 0$ again, we observe that for any $k > 0$, we have the following isomorphism of cohomology groups,

$$H^{r-1+i}(\mathbb{P}(\mathcal{E}^\vee), \mathcal{O}_{\mathbb{P}(\mathcal{E}^\vee)}(-mk-l) \otimes \pi'^* \mathcal{O}((m-l)A)) \\ \cong H^i(\mathbb{P}(\mathcal{E}), \mathcal{O}_{\mathbb{P}(\mathcal{E})}(mk+l-r) \otimes \pi^*(\mathcal{O}((m-l)A) \otimes \det \mathcal{E})),$$

for $i, m > 0$. The latter term vanishes for $m \gg 0$. This says that $\mathcal{O}_{\mathbb{P}(\mathcal{E}^\vee)}(-k) \otimes \mathcal{O}(A)$ is $(r-1)$ -ample for any $k > 0$, which means $\mathcal{O}_{\mathbb{P}(\mathcal{E}^\vee)}(-1)$ is $(r-1)$ -almost ample. \square

The augmented base locus gives us another measure how far a divisor is being ample.

Definition 4.3 (Augmented base locus [ELM⁺06, Definition 1.2]). The augmented base locus of an \mathbf{R} -divisor D on X is the Zariski-closed subset:

$$\mathbf{B}_+(D) = \bigcap_{D=A+E} \mathrm{Supp} E$$

where the intersection is taken over all decompositions $D = A + E$ such that A is ample and E is effective.

Proposition 4.4 (Positivity of normal bundle v.s. positivity of exceptional divisor). *Let $Y \subset X$ be a subscheme of codimension r . Then the normal bundle of the exceptional divisor E in $\mathrm{Bl}_Y X$, $\mathcal{O}_E(E)$ is $(r-1)$ -almost ample if and only if $E \subset \mathrm{Bl}_Y X$ is $(r-1)$ -almost ample.*

Proof. The "if" part of the statment is clear, since restriction of a q -ample divisor to a subscheme is always q -ample. For the "only if" part, observe that $\mathbf{B}_+(E + \epsilon A) \subseteq \text{Supp } E$ for $0 < \epsilon \ll 1$, where A is an ample divisor on X . Now we can apply Brown's theorem [Bro12, Theorem 1.1] to $E + \epsilon A$, which says that an \mathbf{R} -divisor D is q -ample if and only if $D|_{\mathbf{B}_+(D)}$ is q -ample. \square

Definition 4.5 (Nef subscheme). Let Y be a closed subscheme of codimension r of X , a projective scheme, and let E be the exceptional divisor in $\text{Bl}_Y X$. Then we say that Y is *nef* if $\mathcal{O}_E(E)$ is $(r - 1)$ -almost ample.

Remark. Proposition 4.4 says that Y is a nef subscheme if and only if E lies in the closure of the $(r - 1)$ -ample cone of X . If Y is l.c.i. in X , then Y is nef if and only if the normal bundle $\mathcal{N}_{Y/X}$ is nef (proposition 4.2). The advantage of making this more general definition, without requiring Y to be lci, is to include more subschemes that are apparently "positive", for example, a closed point that is not necessarily nonsingular, or if Y is a smooth subvariety with nef normal bundle, the subscheme of X defined by a power of ideal sheaf of Y is also considered as nef in this definition.

The following proposition is the direct generalization of [Ott12, Proposition 3.4].

Proposition 4.6 (Equidimensionality of nef subschemes). *Suppose Y is a nef subscheme of X . Then the restriction of the blowup morphism to E , $\pi|_E : E \rightarrow Y$, is equidimensional. In particular, Y is pure dimensional.*

Proof. Suppose $Y \subset X$ has codimension r . Let $y \in Y$ be a closed point, we want to show $Z := \pi^{-1}(y)$ is of dimension $(r - 1)$. Note that E has dimension $n - 1$, where $n = \dim X$. This implies $\dim Z \geq r - 1$.

On the other hand, $-E$ is π -ample. In particular, $(-E - \epsilon A)|_Z$ is ample for $1 \gg \epsilon > 0$, where A is an ample divisor on E . We also know that $\mathcal{O}_E(E + \epsilon A)$ is $(r - 1)$ -ample, for $1 \gg \epsilon > 0$. By theorem 2.6, this forces Z to have dimension $(r - 1)$. \square

Proposition 4.7 (Pullback of nef subschemes). *Suppose Y is a nef subscheme of X of codimension r , $p : X' \rightarrow X$ a morphism from an equidimensional projective scheme X' . If $p^{-1}(Y)$ has codimension r in X' , then $p^{-1}(Y)$ is nef in X' . In particular, if p is equidimensional, $p^{-1}(Y)$ is nef.*

Proof. We have the following commutative diagram:

$$\begin{array}{ccc} \text{Bl}_{p^{-1}(Y)}(X') & \xrightarrow{\tilde{p}} & \text{Bl}_Y(X) \\ \downarrow & & \downarrow \\ X' & \xrightarrow{p} & X, \end{array}$$

with \tilde{p} induced by the universal property of blowup and $\tilde{p}^*(E) = E'$, where E and E' are exceptional divisors in the respective blowups. We can now apply proposition 2.8 to conclude the proof. \square

Proposition 4.8. *Let Y be an ample (resp. nef) subscheme of codimension r of X . Let Z be a closed subscheme of X . If $Y \cap Z$ has codimension r in Z , then $Y \cap Z$ is an ample (resp. nef) subscheme of Z .*

Proof. Indeed, we have the following commutative diagram

$$\begin{array}{ccc} \mathrm{Bl}_{Y \cap Z} Z & \hookrightarrow & \mathrm{Bl}_Y X \\ \pi_Z \downarrow & & \downarrow \pi_X \\ Z & \hookrightarrow & X. \end{array}$$

Note that the exceptional divisor of π_Z is the restriction of the exceptional divisor E of π_X . If E is $(r-1)$ -ample (resp. $(r-1)$ -almost ample), so is $E|_{\mathrm{Bl}_{Y \cap Z} Z}$. \square

The following theorem generalizes the transitivity property of ample subschemes [Ott12, Proposition 6.4] in the sense that we do not require Y (resp. Z) to be lci in X (resp. Y). This gives further evidence that Ottem's definition of an ample subscheme is a natural one. First, we need a lemma:

Lemma 4.9. *Let X be a projective scheme and let Y be a closed subscheme of X of codimension r . Suppose the blowup of X along Y , $\pi : \mathrm{Bl}_Y X \rightarrow X$, has fiber dimension at most $r-1$. If a line bundle \mathcal{L} on X is q -ample on Y , and for if any $l \geq 0$*

$$H^i(\mathrm{Bl}_Y X, \pi^*(\mathcal{L}^{\otimes m} \otimes \mathcal{O}_X(-l))) = 0$$

for $i > q+r$ and $m \gg 0$, then \mathcal{L} is $(q+r)$ -ample. Here $\mathcal{O}_X(1)$ is an ample line bundle on X .

Proof. Applying the Leray spectral sequence, we have

$$E_2^{p,s} = H^p(X, R^s \pi_* \mathcal{O}_{\mathrm{Bl}_Y X} \otimes \mathcal{L}^{\otimes m} \otimes \mathcal{O}_X(-l)) \Rightarrow H^{p+s}(\mathrm{Bl}_Y X, \pi^*(\mathcal{L}^{\otimes m} \otimes \mathcal{O}_X(-l))).$$

Since the fiber dimension of π is at most $r-1$ [Ott12, Proposition 3.4], $R^s \pi_* \mathcal{O}_{\mathrm{Bl}_Y X} = 0$ and $E_2^{p,s} = 0$ for $s > r-1$.

For $s > 0$, $R^s \pi_* \mathcal{O}_{\mathrm{Bl}_Y X}$ is a coherent sheaf on Y . Indeed, this follows by considering the long exact sequence

$$\cdots \rightarrow R^s \pi_* \mathcal{O}_{\mathrm{Bl}_Y X}(-jE) \rightarrow R^s \pi_* \mathcal{O}_{\mathrm{Bl}_Y X}((-j+1)E) \rightarrow R^s \pi_* \mathcal{O}_E((-j+1)E) \rightarrow \cdots,$$

where E is the exceptional divisor, and the fact that $R^s \pi_* \mathcal{O}_{\tilde{X}}(-jE) = 0$ for $j \gg 0$, since $-E$ is π -ample.

By the q -ampleness of $\mathcal{L}|_Y$, we have $E_2^{p,s} = H^p(X, R^s \pi_* \mathcal{O}_{\mathrm{Bl}_Y X} \otimes \mathcal{L}^{\otimes m} \otimes \mathcal{O}_X(-l)) = 0$ for $p > q$, $s > 0$ and $m \gg 0$.

These two vanishing results imply that $E_h^{p-h, h-1} = E_2^{p-h, h-1} = 0$ for $h \geq 2$, $p > q+r$ and $m \gg 0$.

By the hypothesis,

$$E_\infty^{p,0} = H^p(\mathrm{Bl}_Y X, \pi^*(\mathcal{L}^{\otimes m} \otimes \mathcal{O}_X(-l))) = 0$$

for $p > q+r$ and $m \gg 0$. Hence we arrive at the desired vanishing $E_2^{p,0} = H^p(X, \mathcal{L}^{\otimes m} \otimes \mathcal{O}_X(-l)) = 0$ for $p > q+r$ and $m \gg 0$. \square

Theorem 4.10 (Transitivity of ample subschemes). *Let $Y \subset X$ be an ample subscheme of codimension r_1 , $Z \subset Y$ be an ample subscheme of codimension r_2 . Then $Z \subset X$ is also an ample subscheme of codimension $r_1 + r_2$.*

Lemma 4.11. *Under the same hypothesis as in the theorem, we have the following commutative diagram.*

$$\begin{array}{ccccc}
 \mathrm{Bl}_{\mathcal{I}_Y \cdot \mathcal{I}_Z} X & & & & \\
 \swarrow \pi_Z & \searrow \pi_Y & & & \\
 & \mathrm{Bl}_{\mathcal{I}_Y} X \times_X \mathrm{Bl}_{\mathcal{I}_Z} X^p & \longrightarrow & \mathrm{Bl}_{\mathcal{I}_Z} X & \\
 & \downarrow q & & \downarrow \pi'_Z & \\
 & \mathrm{Bl}_{\mathcal{I}_Y} X & \xrightarrow{\pi'_Y} & X &
 \end{array}$$

Here π'_Z (resp. π'_Y) is the blowup of X along \mathcal{I}_Z (resp. \mathcal{I}_Y), with exceptional divisor E'_Z (resp. E'_Y); π_Z and π_Y are blowups along the ideal sheaves $\mathcal{I}_Z \cdot \mathcal{O}_{\mathrm{Bl}_{\mathcal{I}_Y \cdot \mathcal{I}_Z} X}$ and $\mathcal{I}_Y \cdot \mathcal{O}_{\mathrm{Bl}_{\mathcal{I}_Y \cdot \mathcal{I}_Z} X} \otimes \mathcal{O}_{\mathrm{Bl}_{\mathcal{I}_Y \cdot \mathcal{I}_Z} X}(E'_Z)$, with exceptional divisor E_Z and E_Y respectively. The composition $\pi_Y \circ \pi'_Z = \pi_Z \circ \pi'_Y$ is the blowup map of X along $\mathcal{I}_Y \cdot \mathcal{I}_Z$. The square in the above diagram is a fiber diagram, with ι induced by the maps π_Z and π_Y . Moreover,

- (1) $\pi_Y^* E'_Z = E_Z$ and $\pi_Z^* E'_Y = E_Y + E_Z$.
- (2) ι is a closed immersion.

Proof of lemma. First, let us check that the blowup of X along $\mathcal{I}_Y \cdot \mathcal{I}_Z$ factors through the maps π'_Y and π'_Z . By the universal property of blowup, it suffices to check that the inverse image ideal sheaves $\mathcal{I}_Y \cdot \mathcal{O}_{\mathrm{Bl}_{\mathcal{I}_Y \cdot \mathcal{I}_Z} X}$ and $\mathcal{I}_Z \cdot \mathcal{O}_{\mathrm{Bl}_{\mathcal{I}_Y \cdot \mathcal{I}_Z} X}$ are invertible. Let \mathcal{J} be the inverse of $(\mathcal{I}_Y \cdot \mathcal{I}_Z) \cdot \mathcal{O}_{\mathrm{Bl}_{\mathcal{I}_Y \cdot \mathcal{I}_Z} X}$, i.e. the fractional ideal sheaf such that $(\mathcal{I}_Y \cdot \mathcal{I}_Z) \cdot \mathcal{O}_{\mathrm{Bl}_{\mathcal{I}_Y \cdot \mathcal{I}_Z} X} \cdot \mathcal{J} = \mathcal{O}_{\mathrm{Bl}_{\mathcal{I}_Y \cdot \mathcal{I}_Z} X}$. We check locally that $\mathcal{I}_Y \cdot \mathcal{O}_{\mathrm{Bl}_{\mathcal{I}_Y \cdot \mathcal{I}_Z} X}$ is invertible. Let \mathfrak{a} and \mathfrak{b} be the stalk of $\mathcal{I}_Y \cdot \mathcal{O}_{\mathrm{Bl}_{\mathcal{I}_Y \cdot \mathcal{I}_Z} X}$ and $(\mathcal{I}_Z \cdot \mathcal{O}_{\mathrm{Bl}_{\mathcal{I}_Y \cdot \mathcal{I}_Z} X}) \cdot \mathcal{J}$ at a scheme-theoretic point $x \in \mathrm{Bl}_{\mathcal{I}_Y \cdot \mathcal{I}_Z} X$ respectively, and let $R = \mathcal{O}_{\mathrm{Bl}_{\mathcal{I}_Y \cdot \mathcal{I}_Z} X, x}$ be the local ring at x . Since $\mathfrak{a} \cdot \mathfrak{b} = R$. We may write $\sum_i a_i b_i = 1$, where $a_i \in \mathfrak{a}$ and $b_i \in \mathfrak{b}$. Note that each $a_i b_i \in R$, so there must be some j such that $a_j b_j$ is a unit. Let $u = (a_j b_j)^{-1}$. Let $f : R \rightarrow \mathfrak{a}$ be the R -module homomorphism that sends $r \mapsto r a_j$. We shall see that f is an isomorphism. For any $a \in \mathfrak{a}$, we can write $a = (a_j u) a_j$. Note that $(a_j u) \in R$. Thus, f is onto. Suppose there is an $r \in R$ such that $f(r) = r a_j = 0$. Then $r = r(a_j b_j u) = 0$. Therefore, f is injective. We conclude that $\mathcal{I}_Y \cdot \mathcal{O}_{\mathrm{Bl}_{\mathcal{I}_Y \cdot \mathcal{I}_Z} X}$ is locally free of rank 1, hence is invertible. Applying a similar argument, we see that $\mathcal{I}_Z \cdot \mathcal{O}_{\mathrm{Bl}_{\mathcal{I}_Y \cdot \mathcal{I}_Z} X}$ is also invertible. This gives us the maps π_Z and π_Y .

Next, let us check that $\mathcal{I}_Y \cdot \mathcal{O}_{\mathrm{Bl}_{\mathcal{I}_Z} X} \otimes \mathcal{O}_{\mathrm{Bl}_{\mathcal{I}_Z} X}(E'_Z)$ is an ideal sheaf. Indeed, we have the inclusion $\mathcal{I}_Y \cdot \mathcal{O}_{\mathrm{Bl}_{\mathcal{I}_Z} X} \subset \mathcal{I}_Z \cdot \mathcal{O}_{\mathrm{Bl}_{\mathcal{I}_Z} X} \cong \mathcal{O}_{\mathrm{Bl}_{\mathcal{I}_Z} X}(-E'_Z)$. We then tensor the terms in the inclusion by $\mathcal{O}_{\mathrm{Bl}_{\mathcal{I}_Z} X}(E'_Z)$ to see that $\mathcal{I}_Y \cdot \mathcal{O}_{\mathrm{Bl}_{\mathcal{I}_Z} X} \otimes \mathcal{O}_{\mathrm{Bl}_{\mathcal{I}_Z} X}(E'_Z) \subset \mathcal{O}_{\mathrm{Bl}_{\mathcal{I}_Z} X}$. Applying the universal property of blowup again, we see that $\pi_Z : \mathrm{Bl}_{\mathcal{I}_Y \cdot \mathcal{I}_Z} X \rightarrow \mathrm{Bl}_{\mathcal{I}_Y} X$ and $\pi_Y : \mathrm{Bl}_{\mathcal{I}_Y \cdot \mathcal{I}_Z} X \rightarrow \mathrm{Bl}_{\mathcal{I}_Z} X$ are the same as the blowup of $\mathrm{Bl}_{\mathcal{I}_Y} X$ and $\mathrm{Bl}_{\mathcal{I}_Z} X$ along $\mathcal{I}_Z \cdot \mathcal{O}_{\mathrm{Bl}_{\mathcal{I}_Y} X}$ and $\mathcal{I}_Y \cdot \mathcal{O}_{\mathrm{Bl}_{\mathcal{I}_Z} X} \otimes \mathcal{O}_{\mathrm{Bl}_{\mathcal{I}_Z} X}(E'_Z)$.

For 1, note that $\mathcal{I}_Z \cdot \mathcal{O}_{\mathrm{Bl}_{\mathcal{I}_Z} X} \cong \mathcal{O}_{\mathrm{Bl}_{\mathcal{I}_Z} X}(-E'_Z)$. Therefore we have the surjection $\pi_Y^* \mathcal{O}_{\mathrm{Bl}_{\mathcal{I}_Z} X}(-E'_Z) \rightarrow (\mathcal{I}_Z \cdot \mathcal{O}_{\mathrm{Bl}_{\mathcal{I}_Z} X}) \cdot \mathcal{O}_{\mathrm{Bl}_{\mathcal{I}_Y \cdot \mathcal{I}_Z} X} \cong \mathcal{I}_Z \cdot \mathcal{O}_{\mathrm{Bl}_{\mathcal{I}_Y \cdot \mathcal{I}_Z} X} \cong \mathcal{O}_{\mathrm{Bl}_{\mathcal{I}_Y \cdot \mathcal{I}_Z} X}(-E_Z)$. This is also an injection, since the pullback of a local generator of $\mathcal{I}_Z \cdot \mathcal{O}_{\mathrm{Bl}_{\mathcal{I}_Z} X}$ to $\mathrm{Bl}_{\mathcal{I}_Y \cdot \mathcal{I}_Z} X$ is not a zero divisor, thanks to the fact that $\mathcal{I}_Z \cdot \mathcal{O}_{\mathrm{Bl}_{\mathcal{I}_Y \cdot \mathcal{I}_Z} X}$ is invertible. A similar argument leads to the second statement in 1.

For 2, let W be the scheme-theoretic image of $\mathrm{Bl}_{\mathcal{I}_Y \cdot \mathcal{I}_Z} X$ under ι . It suffices to show that $\mathcal{I}_Y \cdot \mathcal{O}_W$ (resp. $\mathcal{I}_Z \cdot \mathcal{O}_W$) is invertible. Note that the natural surjection $q^* \mathcal{O}_{\mathrm{Bl}_{\mathcal{I}_Y} X}(-E'_Y) \rightarrow \mathcal{I}_Y \cdot \mathcal{O}_W$ is injective if and only if the pullback of a local generator of $\mathcal{O}_{\mathrm{Bl}_{\mathcal{I}_Y} X}(-E'_Y)$ is not

a zero divisor, which follows from the fact that the natural map $\mathcal{O}_W \rightarrow \mathcal{O}_{\text{Bl}_{\mathcal{I}_Y \cdot \mathcal{I}_Z} X}$ is an injection [TS16, Lemma 28.6.3]. We can use the same argument to show that $\mathcal{I}_Z \cdot \mathcal{O}_W$ is also invertible. \square

Proof of theorem. Note that π_Y has fiber dimension at most $r_1 - 1$. This follows from 2 of lemma 4.11 and the fact that π'_Y has fiber dimension at most $r_1 - 1$ (proposition 4.6).

Let \tilde{Y} be the strict transform of Y in $\text{Bl}_{\mathcal{I}_Z} X$. Since Z is an ample subscheme of Y , $E'_Z|_{\tilde{Y}}$ is $(r_2 - 1)$ -ample.

By lemma 4.9, it suffices to prove that given any $l \in \mathbb{Z}_{\geq 0}$,

$$H^i(\text{Bl}_{\mathcal{I}_Y \cdot \mathcal{I}_Z} X, \mathcal{O}_{\text{Bl}_{\mathcal{I}_Y \cdot \mathcal{I}_Z} X}(mE_Z) \otimes \pi_Y^*(\mathcal{O}_{\text{Bl}_{\mathcal{I}_Z} X}(-lH))) = 0$$

for $i > r_1 + r_2 - 1$ and $m \gg 0$. Here H is an ample divisor on $\text{Bl}_{\mathcal{I}_Z} X$. We fix an $l \in \mathbb{Z}_{\geq 0}$ from now on.

Claim 1. $(E_Z - \delta E_Y)|_{E_Y}$ is $(r_2 - 1)$ -ample for $0 < \delta \ll 1$.

Proof of claim. Since $-E_Y$ is π_Y -ample, $(\pi_Y^* E'_Z - \delta E_Y)|_{E_Y} = (E_Z - \delta E_Y)|_{E_Y}$ is $(r_2 - 1)$ -ample, for $0 < \delta \ll 1$, by proposition 2.8. \square

Claim 2. $E_Z + E_Y - \epsilon E_Z$ is $(r_1 - 1)$ -ample for $0 < \epsilon \ll 1$.

Proof of claim. Indeed, $E_Z + E_Y = \pi_Y^* E'_Y$ and E'_Y is $(r_1 - 1)$ -ample by ampleness of $Y \subset X$. Note that $-E_Z$ is π_Y -ample. The claim then follows from proposition 2.8. \square

By the above claims, we may choose a big enough $k \in \mathbb{Z}$ such that $(kE_Z - E_Y)|_{E_Y}$ is $(r_2 - 1)$ -ample and $kE_Z + (k + 1)E_Y$ is $(r_1 - 1)$ -ample.

Write

$$m_1 E_Y + m_2 E_Z = \lambda_1 (kE_Z - E_Y) + \lambda_2 (kE_Z + (k + 1)E_Y) + j_1 E_Y + j_2 E_Z,$$

where $\lambda_2 = \lfloor \frac{m_1 + \lfloor \frac{m_2}{k} \rfloor}{k+2} \rfloor$; $\lambda_1 = \lfloor \frac{m_2}{k} \rfloor - \lambda_2$; $j_1 = ((m_1 + \lfloor \frac{m_2}{k} \rfloor) \bmod (k+2))$ and $j_2 = (m_2 \bmod k)$. Note that $0 \leq j_1 < k + 2$ and $0 \leq j_2 < k$. The precise formulae for λ_1 and λ_2 are not very important. The plan is to choose a big m_2 , then let m_1 increases. As m_1 grows, λ_1 decreases and λ_2 increases. We then use the positivity of $(kE_Z - E_Y)|_{E_Y}$ and $kE_Z + (k + 1)E_Y$ to prove the required vanishing statement.

Since $kE_Z + (k + 1)E_Y$ is $(r_1 - 1)$ -ample, we may find Λ_2 such that

$$(4.2) \quad H^i(\text{Bl}_{\mathcal{I}_Y \cdot \mathcal{I}_Z} X, \mathcal{O}(\lambda_2 (kE_Z + (k + 1)E_Y) + j_1 E_Y + j_2 E_Z) \otimes \pi_Y^*(\mathcal{O}_{\text{Bl}_{\mathcal{I}_Z} X}(-lH))) = 0$$

for $i > r_1 - 1$, $\lambda_2 \geq \Lambda_2$, $0 \leq j_1 < k + 2$ and $0 \leq j_2 < k$.

Applying theorem 3.9 to the scheme E_Y , there is an Λ'_2 such that

$$H^i(E_Y, \mathcal{O}_{E_Y}(\lambda_1 (kE_Z - E_Y) + \lambda_2 (kE_Z + (k + 1)E_Y) + j_1 E_Y + j_2 E_Z) \otimes \pi_Y^* \mathcal{O}_{\text{Bl}_{\mathcal{I}_Z} X}(-lH)) = 0$$

for $i > (r_2 - 1) + (r_1 - 1)$, $\lambda_1 \geq 0$, $\lambda_2 \geq \Lambda'_2$, $0 \leq j_1 < k + 2$ and $0 \leq j_2 < k$. This implies

$$(4.3) \quad \begin{aligned} H^i(\text{Bl}_{\mathcal{I}_Y \cdot \mathcal{I}_Z} X, \mathcal{O}(m_2 E_Z + m_1 E_Y) \otimes \pi_Y^*(\mathcal{O}_{\text{Bl}_{\mathcal{I}_Z} X}(-lH))) \\ \cong H^i(\text{Bl}_{\mathcal{I}_Y \cdot \mathcal{I}_Z} X, \mathcal{O}(m_2 E_Z + (m_1 + 1)E_Y) \otimes \pi_Y^*(\mathcal{O}_{\text{Bl}_{\mathcal{I}_Z} X}(-lH))) \end{aligned}$$

for $i > r_1 + r_2 - 1$, $0 < m_1 + 1 < (k + 1)\lfloor \frac{m_2}{k} \rfloor + k + 2$ and $\lfloor \frac{m_1 + 1 + \lfloor \frac{m_2}{k} \rfloor}{k+2} \rfloor \geq \Lambda'_2$.

Choose some big M_2 such that $\lfloor \frac{\lfloor \frac{M_2}{k} \rfloor}{k+2} \rfloor \geq \max\{\Lambda_2, \Lambda'_2\}$. Applying (4.3) repeatedly, we have for $m_2 > M_2$,

$$(4.4) \quad H^i(\mathrm{Bl}_{\mathcal{J}_Y \cdot \mathcal{J}_Z} X, \mathcal{O}(m_2 E_Z) \otimes \pi_Y^*(\mathcal{O}_{\mathrm{Bl}_{\mathcal{J}_Z} X}(-lH))) \\ \cong H^i(\mathrm{Bl}_{\mathcal{J}_Y \cdot \mathcal{J}_Z} X, \mathcal{O}(m_2 E_Z + (k+1)\lfloor \frac{m_2}{k} \rfloor E_Y) \otimes \pi_Y^*(\mathcal{O}_{\mathrm{Bl}_{\mathcal{J}_Z} X}(-lH)))$$

for $i > r_1 + r_2 - 1$. The above cohomology group can be rewritten as

$$H^i(\mathrm{Bl}_{\mathcal{J}_Y \cdot \mathcal{J}_Z} X, \mathcal{O}(\lfloor \frac{m_2}{k} \rfloor (kE_Z + (k+1)E_Y) + (m_2 - k\lfloor \frac{m_2}{k} \rfloor)E_Z) \otimes \pi_Y^*(\mathcal{O}_{\mathrm{Bl}_{\mathcal{J}_Z} X}(-lH))),$$

which is 0 by (4.2). \square

We then prove the analogue of theorem 4.10 for nef subschemes. The idea of the proof is essentially the same, although we have to use the full statement of theorem 3.9 by allowing a nef term, as well as take extra care with the variables.

Theorem 4.12 (Transitivity of nef subschemes). *Let $Y \subset X$ be a nef subscheme of codimension r_1 , $Z \subset Y$ be a nef subscheme of codimension r_2 . Then $Z \subset X$ is also a nef subscheme of codimension $r_1 + r_2$.*

Proof. Lemma 4.11 still holds under the hypothesis of the theorem. We shall use the same notation as in lemma 4.11. Since $-E'_Z$ and $-E'_Y$ is π'_Z -ample and π'_Y -ample respectively, we may choose an ample divisor A' on X such that $\pi_Z'^* A' - E'_Z$ and $\pi_Y'^* A' - E'_Y$ are ample. Let $A = \pi_Y^* \pi_Z'^* A'$ be the pullback of A to $\mathrm{Bl}_{\mathcal{J}_Y \cdot \mathcal{J}_Z} X$, note that A is nef.

Note that we can write $kE_Z + A$ as $\pi_Y^*((k+1)E'_Z + (\pi_Z'^* A' - E'_Z))$. By lemma 4.9, it suffices to prove that given any $l \geq 0$, for $k \gg 0$

$$H^i(\mathrm{Bl}_{\mathcal{J}_Y \cdot \mathcal{J}_Z} X, \mathcal{O}(m_2(kE_Z + A)) \otimes \pi_Y^*(\mathcal{O}_{\mathrm{Bl}_{\mathcal{J}_Z} X}(-l))) = 0$$

for $i > r_1 + r_2 - 1$ and $m_2 \gg 0$. We fix l and k from this point on.

Note that $F'_1 := (E_Z + \frac{1}{3k}\pi_Y^*(\pi_Z'^* A' - E'_Z) - \frac{1}{k_1}E_Y)$ is $(r_2 - 1)$ -ample when restricted to E_Y and $F'_2 := E_Z + E_Y + \frac{1}{3k}\pi_Z^*(\pi_Y'^* A' - E'_Y) - \frac{1}{k_1}E_Z$ is $(r_1 - 1)$ -ample for $k_1 \gg 0$. We fix such a k_1 . Let $\alpha = 3kk_1 - k_1$ and $\beta = 3kk_1 - k_1 - 3k$. Let $F_1 = 3kk_1\beta F'_1$ and $F_2 = 3kk_1\alpha F'_2$. They are both integral divisors. In fact, $F_1 = \beta(\alpha E_Z - 3kE_Y + k_1 A)$ and $F_2 = \alpha(\beta E_Z + \alpha E_Y + k_1 A)$.

Write

$$m_1 E_Y + m_2(kE_Z + A) = \lambda_1 F_1 + \lambda_2 F_2 + \lambda_3 A + j_1 E_Y + j_2 E_Z,$$

where $\lambda_2 = \lfloor \frac{m_1 + 3\beta k \lfloor \frac{m_2 k}{\alpha \beta} \rfloor}{\alpha^2 + 3\beta k} \rfloor$; $\lambda_1 = \lfloor \frac{m_2 k}{\alpha \beta} \rfloor - \lambda_2$; $\lambda_3 = m_2 - \lambda_1 \beta k_1 - \lambda_2 \alpha k_1$; $j_1 = ((m_1 + 3\beta k \lfloor \frac{m_2 k}{\alpha \beta} \rfloor) \bmod (\alpha^2 + 3\beta k))$ and $j_2 = (m_2 k \bmod \alpha \beta)$. Note that if $0 \leq m_1 \leq \alpha^2 \lfloor \frac{m_2 k}{\alpha \beta} \rfloor$, then $\lambda_1 \geq 0$ and $\lambda_3 \geq 0$.

Since F_2 is $(r_1 - 1)$ -ample and A is nef, there is a Λ_2 such that

$$(4.5) \quad H^i(\mathrm{Bl}_{\mathcal{J}_Y \cdot \mathcal{J}_Z} X, \mathcal{O}(\lambda_2 F_2 + \lambda_3 A + j_2 E_Z) \otimes \pi_Y^*(\mathcal{O}_{\mathrm{Bl}_{\mathcal{J}_Z} X}(-l))) = 0$$

for $i > r_1 - 1$, $\lambda_2 > \Lambda_2$, $\lambda_3 \geq 0$ and $0 \leq j_2 < \alpha \beta$.

Since $F_1|_{E_Y}$ is $(r_2 - 1)$ -ample, F_2 is $(r_1 - 1)$ -ample and A is nef, there is a Λ'_2 such that

$$H^i(E_Y, \mathcal{O}_{E_Y}(\lambda_1 F_1 + \lambda_2 F_2 + \lambda_3 A + j_1 E_Y + j_2 E_Z) \otimes \pi_Y^*(\mathcal{O}_{\mathrm{Bl}_{\mathcal{J}_Z} X}(-l))) = 0$$

for $i > (r_2 - 1) + (r_1 - 1)$, $\lambda_2 > \Lambda'_2$, $\lambda_1 \geq 0$, $\lambda_3 \geq 0$, $0 \leq j_1 < \alpha^2 + 3\beta k$ and $0 \leq j_2 < \alpha\beta$. This implies if $\lfloor \frac{m_1 + 3\beta k \lfloor \frac{m_2 k}{\alpha\beta} \rfloor}{\alpha^2 + 3\beta k} \rfloor > \Lambda'_2$,

$$\begin{aligned} H^i(\mathrm{Bl}_{\mathcal{I}_Y \cdot \mathcal{I}_Z} X, \mathcal{O}(m_2(kE_Z + A)) \otimes \pi_Y^*(\mathcal{O}_{\mathrm{Bl}_{\mathcal{I}_Z} X}(-l))) \\ \cong H^i(\mathrm{Bl}_{\mathcal{I}_Y \cdot \mathcal{I}_Z} X, \mathcal{O}(\alpha^2 \lfloor \frac{m_2 k}{\alpha\beta} \rfloor E_Y + m_2(kE_Z + A)) \otimes \pi_Y^*(\mathcal{O}_{\mathrm{Bl}_{\mathcal{I}_Z} X}(-l))) \end{aligned}$$

for $i > r_1 + r_2 - 1$.

The above cohomology groups can be rewritten as

$$H^i(\mathrm{Bl}_{\mathcal{I}_Y \cdot \mathcal{I}_Z} X, \mathcal{O}(\lfloor \frac{m_2 k}{\alpha\beta} \rfloor F_2 + \lambda_3 A + j_2 E_Z) \otimes \pi_Y^*(\mathcal{O}_{\mathrm{Bl}_{\mathcal{I}_Z} X}(-l)))$$

where $\lambda_3 \geq 0$ and $0 \leq j_2 < \alpha\beta$. By (4.5), the above cohomology groups vanish for $m_2 \gg 0$. \square

The following corollary says that intersection of 2 ample (resp. nef) subschemes is ample (resp. nef), assuming the intersection has the desired codimension. It is the generalization of [Ott12, Proposition 6.3], in the sense that we do not assume that X is smooth and the subschemes are lci in X .

Corollary 4.13 (Intersection of ample or nef subschemes). *If Y and Z are both ample (resp. nef) subschemes of X , of codimension r and s respectively and $Y \cap Z$ has codimension $r + s$ in X , then $Y \cap Z$ is an ample (resp. nef) subscheme of X .*

Proof. By proposition 4.8, $Y \cap Z$ is an ample (resp. nef) subscheme of Z . We now conclude using the transitivity property of ample (resp. nef) subschemes (theorem 4.10 or theorem 4.12 respectively). \square

5. POSITIVITY OF A LINE BUNDLE UPON RESTRICTION TO AN AMPLE SUBSCHEME

If a line bundle is ample after restricting to an ample subscheme, it is reasonable to expect the line bundle to exhibit some positivity features. The following theorem demonstrates a nice interplay between ample subschemes and q -ample divisors.

Theorem 5.1. *Let X be a projective scheme of dimension n . Let Y be an ample subscheme of X of codimension r . Suppose \mathcal{L} is a line bundle on X , and that its restriction $\mathcal{L}|_Y$ to Y is q -ample. Then \mathcal{L} is $(q + r)$ -ample.*

Proof. We fix an ample line bundle $\mathcal{O}_X(1)$ on X . Let $\pi : \tilde{X} \rightarrow X$ be the blowup of X along Y .

Step 1. *Pass to the blowup.*

By lemma 4.9, it suffices to prove that

$$(5.1) \quad H^i(\tilde{X}, \pi^*(\mathcal{L}^{\otimes m} \otimes \mathcal{O}_X(-l))) = 0$$

for $i > q + r$ and $m \gg 0$.

Step 2. *Pass to the exceptional divisor.*

We claim that it is enough to show that there is an m_0 such that

$$(5.2) \quad H^i(E, \pi^*(\mathcal{L}^{\otimes m} \otimes \mathcal{O}_X(-l)) \otimes \mathcal{O}_E(kE)) = 0$$

for $i > r + q - 1$, $m > m_0$ and $k \geq 1$. Here E is the exceptional divisor on the blowup \tilde{X} .

Indeed, let us consider the short exact sequence:

$$\begin{aligned} 0 \rightarrow \pi^*(\mathcal{L}^{\otimes m} \otimes \mathcal{O}_X(-l)) \otimes \mathcal{O}_{\tilde{X}}((k-1)E) &\rightarrow \pi^*(\mathcal{L}^{\otimes m} \otimes \mathcal{O}_X(-l)) \otimes \mathcal{O}_{\tilde{X}}(kE) \\ &\rightarrow \pi^*(\mathcal{L}^{\otimes m} \otimes \mathcal{O}_X(-l)) \otimes \mathcal{O}_E(kE) \rightarrow 0. \end{aligned}$$

By looking at the long exact sequence of cohomology groups induced from the above short exact sequence and using the hypothesis (5.2), we observe that

$$(5.3) \quad H^i(\tilde{X}, \pi^*(\mathcal{L}^{\otimes m} \otimes \mathcal{O}_X(-l)) \otimes \mathcal{O}_{\tilde{X}}((k-1)E)) \cong H^i(\tilde{X}, \pi^*(\mathcal{L}^{\otimes m} \otimes \mathcal{O}_X(-l)) \otimes \mathcal{O}_{\tilde{X}}(kE))$$

for $i > r + q$, $m > m_0$ and $k \geq 1$.

Since E is $(r-1)$ -ample, for any fixed m ,

$$H^i(\tilde{X}, \pi^*(\mathcal{L}^{\otimes m} \otimes \mathcal{O}_X(-l)) \otimes \mathcal{O}_{\tilde{X}}(kE)) = 0$$

for $k \gg 0$ and $i > r - 1$. Together with the isomorphisms in (5.3), we have the desired vanishing result (5.1).

Step 3. Rewrite the line bundles of interest in (5.2) in terms of q - and $(r-1)$ - ample line bundles.

Note that $-E$ is π -ample, there is an $N > 0$ such that $\pi^*(\mathcal{L}^{\otimes N}) \otimes \mathcal{O}_{\tilde{X}}(-E)$ is q -ample, by proposition 2.8. We can replace \mathcal{L} by $\mathcal{L}^{\otimes N}$ and assume that $\pi^*(\mathcal{L}) \otimes \mathcal{O}_E(-E)$ is q -ample. We now rewrite the line bundle on E in (5.2):

$$\pi^*(\mathcal{L}^{\otimes m} \otimes \mathcal{O}_X(-l)) \otimes \mathcal{O}_E(kE) \cong \pi^*(\mathcal{O}_X(-l)) \otimes \mathcal{O}_E((k+m)E) \otimes (\pi^*(\mathcal{L}) \otimes \mathcal{O}_E(-E))^{\otimes m}$$

with the second term $\mathcal{O}_E((k+m)E)$ on the right hand side being an $(r-1)$ -ample line bundle, and the third term $(\pi^*(\mathcal{L}) \otimes \mathcal{O}_E(-E))^{\otimes m}$ being an q -ample line bundle.

We now apply theorem 3.9 with $\mathcal{L}_1 := \pi^*(\mathcal{L}) \otimes \mathcal{O}_E(-E)$, $\mathcal{L}_2 = \mathcal{O}_E(E)$ and $M_2 = 1$ to conclude. \square

One may ask whether we have a converse to theorem 5.1, i.e., given an r -ample line bundle \mathcal{L} on a projective scheme X , there is a codimension r ample subscheme Y , such that $\mathcal{L}|_Y$ is ample. Demailly, Peternell and Schneider gave a counter-example to this in [DPS96, Example 5.6]:

Example 5.2. Let S be a general quartic surface in \mathbb{P}^3 . Let $X = \mathbb{P}(\Omega_S^1)$. They showed that $-K_X$ is 1-ample, and yet for any ample divisor Y in X , $(-K_X)^2 \cdot Y < 0$, thus $-K_X$ cannot be ample when it is restricted to any ample divisor.

For the reader's convenience, we shall include the proof of $-K_X$ being 1-ample in example [DPS96]. In fact, it might be worthwhile to extract from the argument of [DPS96, Example 5.6] the following general property.

Proposition 5.3. *Let*

$$(5.4) \quad 0 \rightarrow \mathcal{E}' \rightarrow \mathcal{E} \rightarrow \mathcal{L} \rightarrow 0$$

be a short exact sequence of vector bundles on a projective scheme X . We assume \mathcal{E} to be a q -ample vector bundle of rank r , \mathcal{E}' is of rank $(r-1)$ and \mathcal{L} is of rank 1. Then \mathcal{E}' is $(q+1)$ -ample.

Proof. We first dualize (5.4), then take symmetric product, and dualize again. This will give us the following short exact sequence

$$0 \rightarrow \mathrm{Sym}^k \mathcal{E}' \rightarrow \mathrm{Sym}^k \mathcal{E} \rightarrow \mathrm{Sym}^{k-1} \mathcal{E} \otimes \mathcal{L} \rightarrow 0.$$

Fix an ample line bundle $\mathcal{O}_X(1)$ on X , and tensor the above short exact sequence with $\mathcal{O}_X(-l)$, for $l \geq 0$. Note that $H^i(X, \text{Sym}^k \mathcal{E} \otimes \mathcal{O}_X(-l)) = H^i(X, \text{Sym}^{k-1} \mathcal{E} \otimes \mathcal{L} \otimes \mathcal{O}_X(-l)) = 0$, for $i > q$ and $k \gg 0$. Hence $H^i(X, \text{Sym}^k \mathcal{E}' \otimes \mathcal{O}_X(-l)) = 0$, for $i > q + 1$ and $k \gg 0$. \square

Going back to the example 5.2, note that $\Omega_S^1 \cong \mathcal{T}_S$, where \mathcal{T}_S is the tangent sheaf of S . We have the following short exact sequence of locally free sheaves on S .

$$0 \rightarrow \mathcal{T}_S \rightarrow \mathcal{T}_{\mathbb{P}^3|_S} \rightarrow \mathcal{O}_S(S) \rightarrow 0.$$

The tangent bundle of a projective space is ample, therefore the tangent bundle of S is 1-ample by the lemma. Since $\mathcal{O}_X(-K_X) \cong \mathcal{O}_{\mathbb{P}(\Omega_S^1)}(2)$, $-K_X$ is 1-ample. It is not ample since the tangent bundle of S is not ample (S is a K3-surface).

Remark. Interestingly, we note that

$$H^2(X, K_X - K_X) \cong H^2(S, \mathcal{O}_S) \neq 0,$$

Hence, Kodaira-type vanishing theorem fails for $-K_X$, which is 1-ample. Ottem also gave a counterexample to Kodaira-type vanishing theorem for q -ample divisors [Ott12, Chapter 9].

Example 5.4. One may also ask if we can relax the positivity assumption on Y in theorem 5.1. For example, if we only assume that the normal bundle of Y is ample, we shall see the conclusion of the theorem does not hold in general. Let us start with a smooth ample subvariety $Y \subset X$ of a smooth projective variety. We blowup a closed point p in $X \setminus Y$. Observe that the normal bundle of $Y \subset \text{Bl}_p X$ is still ample. Let $E \cong \mathbb{P}^{n-1}$ be the exceptional divisor, and let A be an ample divisor on $\text{Bl}_p(X)$. Then $E + \epsilon A$ is not $(n - 2)$ -ample, for $0 < \epsilon \ll 1$, since it is anti-ample when restricted to the exceptional divisor. But $(E + \epsilon A)|_Y = \epsilon A|_Y$ is ample.

On the other hand, as we shall see in the following section, a small yet interesting part of the theorem still holds if we assume Y is a nef subvariety.

6. RESTRICTION OF A PSEUDOEFFECTIVE DIVISOR TO A NEF SUBVARIETY

There are not many results regarding the positivity of subvariety with nef normal bundle, in terms of intersection theory. Here are two of such results the author is aware of.

In Fulton-Lazarsfeld's work [FL83] (see also [Laz04, Theorem 8.4.1]), they proved that if Y is a closed, lci subvariety of a projective variety X and the normal bundle of Y is nef, then for any closed subscheme $Z \subset X$ with $\dim Y + \dim Z \geq \dim X$, $\deg_H(Y \cdot Z) \geq 0$. (Here H is an ample divisor on X .) On the other hand, it is not hard to show that if Y has globally generated normal bundle, then restriction of any effective cycle to Y is either effective or 0 [Ful98, Theorem 12.1.a)].

We show that the restriction of a pseudoeffective divisor to a nef subvariety is still pseudoeffective.

Theorem 6.1. *Let Y be a nef subvariety of codimension r of a projective variety X . Then*

$$\iota^* \overline{\text{Eff}}^1(X) \subseteq \overline{\text{Eff}}^1(Y)$$

and

$$\iota^* \text{Big}(X) \subseteq \text{Big}(Y).$$

Here $\iota : Y \hookrightarrow X$ is the inclusion map, $\iota^* : N^1(X)_{\mathbf{R}} \rightarrow N^1(Y)_{\mathbf{R}}$ is the induced map on the Néron-Severi group with \mathbf{R} -coefficients and $\overline{\text{Eff}}^1(X)$ (resp. $\text{Big}(X)$) is the cone of pseudoeffective (resp. big) \mathbf{R} -Cartier divisors.

Remark. Before proving the theorem, let us point out it is rather straightforward to obtain the conclusion under the stronger assumptions in theorem 5.1 and the added assumption that X and Y are integral. Let D be a pseudoeffective divisor on X , i.e. $-D$ is not $(n-1)$ -ample (theorem 2.7). Suppose on the contrary $D|_Y$ is not pseudoeffective. Then $-D|_Y$ is $(n-r-1)$ -ample. This gives a contradiction to theorem 5.1.

Proof of theorem 6.1. A divisor is big if and only if it can be written as the sum of a pseudoeffective divisor and an ample divisor. Therefore, we can focus on the pseudoeffective case. We shall follow the steps in the proof of theorem 5.1 closely. Recall that a Cartier divisor D is $(n-1)$ -ample if and only if $-D$ is not pseudoeffective (theorem 2.7). Given a line bundle \mathcal{L} on X such that $\mathcal{L}|_Y$ is $(n-r-1)$ -ample, we need to show \mathcal{L} is $(n-1)$ -ample. Fix an ample line bundle $\mathcal{O}_X(1)$ on X .

Step 1 (Pass to the blowup). *It suffices to show for any $l \geq 0$, there is an m_0 such that $H^n(\tilde{X}, \pi^*(\mathcal{L}^{\otimes m} \otimes \mathcal{O}_X(-l))) = 0$ for $m \geq m_0$.*

This is true by lemma 4.9.

We now fix l .

Step 2 (Pass to the exceptional divisor). *It is enough to show that there is an m_0 such that*

$$H^{n-1}(E, \pi^*(\mathcal{L}^{\otimes m} \otimes \mathcal{O}_X(-l)) \otimes \mathcal{O}_E(kE)) = 0$$

for $m \geq m_0$ and $k \geq 1$.

We just have to repeat the argument in step 2 in the proof of theorem 5.1, i.e. consider the long exact sequence of cohomologies associated to

$$\begin{aligned} 0 \rightarrow \pi^*(\mathcal{L}^{\otimes m} \otimes \mathcal{O}_X(-l)) \otimes \mathcal{O}_{\tilde{X}}((k-1)E) &\rightarrow \pi^*(\mathcal{L}^{\otimes m} \otimes \mathcal{O}_X(-l)) \otimes \mathcal{O}_{\tilde{X}}(kE) \\ &\rightarrow \pi^*(\mathcal{L}^{\otimes m} \otimes \mathcal{O}_X(-l)) \otimes \mathcal{O}_E(kE) \rightarrow 0. \end{aligned}$$

Also note that for a fixed m ,

$$H^n(\tilde{X}, \pi^*(\mathcal{L}^{\otimes m} \otimes \mathcal{O}_X(-l)) \otimes \mathcal{O}_{\tilde{X}}(kE)) = 0$$

for $k \gg 0$. Indeed, E is $(n-1)$ -ample ($-E$ is not pseudoeffective!).

Step 3 (Rewrite in terms of an $(n-r-1)$ -ample line bundle and an $(r-1)$ -almost ample line bundle).

Replacing \mathcal{L} with $\mathcal{L}^{\otimes N}$ for N large enough, we may assume $\pi^*\mathcal{L} \otimes \mathcal{O}_E(-E)$ is $(n-r-1)$ -ample, by proposition 2.8. Now we can write

$$\pi^*(\mathcal{L}^{\otimes m} \otimes \mathcal{O}_X(-l)) \otimes \mathcal{O}_E(kE) \cong \pi^*\mathcal{O}_X(-l) \otimes (\pi^*\mathcal{L} \otimes \mathcal{O}_E(-E))^{\otimes m} \otimes \mathcal{O}_E((k+m)E).$$

By proposition 3.10, there is an m_0 such that

$$H^{n-1}(E, \pi^*\mathcal{O}_X(-l) \otimes (\pi^*\mathcal{L} \otimes \mathcal{O}_E(-E))^{\otimes m} \otimes \mathcal{O}_E((k+m)E)) = 0$$

for $k \geq 1$ and $m \geq m_0$. This proves the theorem. \square

Remark. Suppose the conclusion of theorem 6.1 holds, the normal bundle of Y is not necessarily nef. Take a 3-fold with Picard number 1 that contains a rational curve C with normal bundle $\mathcal{O}(-1) \oplus \mathcal{O}(-1)$. The condition $D \cdot C > 0$ for any pseudoeffective divisor D is obvious due to the Picard number 1 condition on the 3-fold. This example is taken from Ottem's paper [Ott, Example 1.2.vii].

Boucksom, Demailly, Păun and Peternell showed that the dual cone of the pseudoeffective cone is the cone of movable curves [BDPP13]. Hence we have the equivalent statement:

Corollary 6.2. *With the same assumptions as in theorem 6.1, the map on the numerical equivalence classes of 1-cycles, $\iota_* : N_1(Y) \rightarrow N_1(X)$, induces $\iota_* : \overline{\text{Mov}}_1(Y) \rightarrow \overline{\text{Mov}}_1(X)$, where $\overline{\text{Mov}}_1(Y)$ and $\overline{\text{Mov}}_1(X)$ are the cones of movable curves in Y and X respectively.*

We apply the adjunction formula to get

Corollary 6.3. *If both X and Y are non-singular, Y has nef normal bundle and K_X is pseudoeffective, then K_Y is also pseudoeffective. If K_X is big, then K_Y is also big.*

Remark. The first assertion in the above corollary follows also from [BDPP13] and the theory of deformation of rational curve. More specifically, Boucksom-Demailly-Păun-Peternell showed that on a smooth projective variety Z , K_Z is pseudoeffective if and only if Z is not uniruled. If Y is uniruled, take a smooth rational curve C that covers Y . By considering the short exact sequence of normal bundles on C , we see that the normal bundle of C in X is nef. Thus, X is uniruled.

7. WEAKLY MOVABLE CONE

We shall define and study the weakly movable cone. In this section, we assume the ground field k is algebraically closed and of characteristic zero. On a smooth projective variety, we know that the movable cone of divisors is the smallest closed convex cone that contain all the pushforwards of nef divisors from X_π , where $\pi : X_\pi \rightarrow X$ is projective and birational. With this in mind, we define the weakly movable cone as the closure of the cone that is generated by pushforward of cycles of nef subvariety via generically finite morphism. We find that it contains the movable cone and satisfies some desirable intersection theoretic properties.

First, let us recall the definition of a family of effective cycles. We shall follow Fulger-Lehmann's definition [FL].

Definition 7.1 (Family of effective cycles). Let X be a projective variety over k . A *family of effective d -cycles* on X with \mathbb{Z} -coefficient, $(g : U \rightarrow W)$, consists of a closed reduced subscheme $\text{Supp } U$ of $W \times_k X$, where W is a variety over k ; a coefficient $a_i \in \mathbb{Z}_{>0}$ for each irreducible component U_i of $\text{Supp } U$; and the projection morphisms $g_i : U_i \rightarrow W$ is proper and flat of relative dimension d .

Over a closed point $w \in W$, $g_i^{-1}(w)$ is a closed subscheme of X . Its fundamental cycle $[g_i^{-1}(w)]$ is a d -cycle of X . We define the *cycle theoretic fiber* over w to be $\sum a_i [g_i^{-1}(w)]$.

We say that the family of effective d -cycles is *irreducible* if $\text{Supp } U$ is irreducible.

Remark. Kollár's definition [Kol96, Definition I.3.11] of a *well-defined family of d -dimensional proper algebraic cycles* is more general. By [Kol96, Lemma I.3.14], given an effective, well-defined family of proper algebraic cycles of a projective variety X over a variety W (both are over k), there is a proper surjective morphism $W' \rightarrow W$ from a variety W' such that there is a family of effective cycles (in the sense of Fulger-Lehmann) over W' that "preserves" the cycle theoretic fibers over the closed points of the original family. Therefore for our purpose, it is enough to use Fulger-Lehmann's definition.

Definition 7.2 (Strictly movable cycles [FL, Definition 3.1]). We say that a family of effective d -cycles of X $(g : U \rightarrow W)$ is *strictly movable* if each of the irreducible component U_i of $\text{Supp } U$ dominates X via the second projection.

We say that an effective d -cycle of X (with \mathbb{Z} -coefficient) is *strictly movable* if it is the cycle theoretic fiber over a closed point of a strictly movable family of d -cycles on X .

We define the *movable cone of d -cycles* $\overline{\text{Mov}}_d(X) \subset N_d(X)$ to be the closure of the convex cone generated by strictly movable d -cycles.

Proposition 7.3. *The movable cone of d -cycles is the closure of the convex cone generated by irreducible, strictly movable d -cycles.*

Proof. Suppose $\sum a_i Z_i$ is the cycle theoretic fiber over a closed point of a family of strictly movable d -cycles $(g : U \rightarrow W)$ with irreducible components U_i . It suffices to show that Z_i is algebraically equivalent to a sum of irreducible strictly movable d -cycles. If the generic fiber of $p_i : U_i \rightarrow W$ is geometrically integral, then the fiber over a general closed point is also (geometrically) integral [Gro66, Thorne 9.7.7], and we are done.

Suppose the generic fiber of p_i is not geometrically integral. Let η_W be the generic point of W , let $\overline{k(\eta_W)}$ be the algebraic closure of $k(\eta_W)$ and let $U'_{ij} \subset X_{\overline{k(\eta_W)}}$ be the irreducible components of $\text{Spec } \overline{k(\eta_W)} \times_{\text{Spec } k(\eta_W)} U_i$. We may take a finite field extension $k(\eta_W) \subset K$, such that the generators of the ideal sheaves of U'_{ij} are defined over K . Then all the irreducible components of $\text{Spec } K \times_{\text{Spec } k(\eta_W)} U_i$ are geometrically integral. These components dominate the generic fiber of p_i . Take a variety V with function field K such that the map $\text{Spec } K \rightarrow \text{Spec } k(\eta_W)$ extends to $V \rightarrow W$. By generic flatness, we may replace V by a smaller open set and assume that each irreducible components U_{ij} of $V \times_W U_i$ is flat over V . Note that all U_{ij} dominates U_i , hence also X . Thus, each U_{ij} is a strictly movable family of d -cycles of X over V (with coefficient 1), and the cycle theoretic fiber over a general closed point of V is (geometrically) integral, by [Gro66, Thorne 9.7.7] again. Then Z_i is algebraically equivalent to the sum of the cycle theoretic fibers of U_{ij} 's, with \mathbb{Z} -coefficient, over a general closed point of V . \square

Proposition 7.4. *An irreducible, strictly movable cycle can be realized as the pushforward of a multiple of the cycle class of a nef subvariety via a proper, surjective morphism, up to numerical equivalence.*

Proof. From the proof of proposition 7.3, we may assume the irreducible, strictly movable cycle is the cycle theoretic fiber over a closed point of an irreducible, strictly movable family of $(g : U \rightarrow W)$, with the fiber of $g' : \text{Supp } U \rightarrow W$ over a general closed point of W integral. Using the argument in [FL, Remark 2.13] or [Kol96, Proposition I.3.14], we may assume W is projective. We note that a closed point $w \in W$ is nef, hence $g'^{-1}(w)$ is also nef, by proposition 4.7, and that $g'^{-1}(w)$ is integral if w is general. \square

Definition 7.5 (Weakly movable cone). Let X be a projective variety over k . We define the *weakly movable cone* $\overline{\text{WMov}}_d(X) \in N_d(X)$ to be the closure of the convex cone generated by $\pi_*[Z]$, where $\pi : Y \rightarrow X$ is proper, surjective morphism from a projective variety and Z is a nef subvariety of dimension d in Y .

We shall compare the movable cone and the weakly movable cone.

Proposition 7.6. *Let X be a projective variety over k . We have*

$$\overline{\text{Mov}}_d(X) \subseteq \overline{\text{WMov}}_d(X).$$

In particular, $\overline{\text{WMov}}_d(X)$ is a full dimensional cone in $N_d(X)$.

Proof. This follows from proposition 7.4 and [FL, Proposition 3.8]. \square

The following proposition is an analogue of the first statement of [FL, Lemma 3.6].

Proposition 7.7. *Let X' and X be projective variety over k . Suppose $h : X' \rightarrow X$ is a proper surjective morphism. Then $h_* \overline{\text{WMov}}_d(X') \subseteq \overline{\text{WMov}}_d(X)$.*

Proof. It follows from the definition of the weakly movable cone. \square

The following theorem is an analogue of [FL, Lemma 3.10].

Theorem 7.8. *Let X be a projective variety over k and let $\alpha \in \overline{\text{WMov}}_d(X)$. Then*

- (1) *If $\beta \in \overline{\text{Eff}}^1(X)$, then $\beta \cdot \alpha \in \overline{\text{Eff}}_{d-1}(X)$.*
- (2) *Let H be a big Cartier divisor. If $H \cdot \alpha = 0$, then $\alpha = 0$.*
- (3) *If $\beta \in \text{Nef}^1(X)$ then $\beta \cdot \alpha \in \overline{\text{WMov}}_{d-1}(X)$.*

Proof. For (1), we may assume $\alpha = \pi_*[Z]$, where $\pi : Y \rightarrow X$ is a proper, surjective map and Z a nef subvariety of Y . By projection formula, we have $\beta \cdot \pi_*[Z] = \pi_*(\pi^*\beta \cdot [Z])$. We know that $\pi^*\beta$ is pseudoeffective. By theorem 6.1, $\pi^*\beta \cdot [Z] \in \overline{\text{Eff}}_{d-1}(Y)$. Since $\pi_*\overline{\text{Eff}}_{d-1}(Y) \subseteq \overline{\text{Eff}}_{d-1}(X)$, we have $\beta \cdot \pi_*[Z] \in \overline{\text{Eff}}_{d-1}(X)$.

For (2), we follow Fulger-Lehmann's argument [FL, Proof of Lemma 3.10]. We write $H = A + E$, where A is ample and E is effective. By (1), $A \cdot \alpha, E \cdot \alpha \in \overline{\text{Eff}}_{d-1}(X)$. In particular, $H \cdot \alpha = 0$ implies $A \cdot \alpha = 0$ [FLb, Corollary 3.8], which can only happen when $\alpha = 0$ [FLb, Corollary 3.16].

For (3), we may again assume $\alpha = \pi_*[Z]$, where $\pi : Y \rightarrow X$ is a proper, surjective map and Z a nef subvariety of Y . We also assume $d \geq 2$, otherwise the result already follows from (1). Note that $\pi_*\overline{\text{WMov}}_{d-1}(Y) \subseteq \overline{\text{WMov}}_{d-1}(X)$ by the definition of weakly movable cone. It suffices to show that $H \cdot [Z] \in \overline{\text{WMov}}_{d-1}(Y)$, where H is a very ample divisor on Y . We may assume that $H \cap Z$ is of dimension $d - 1$ and is integral [Jou83, Corollaire 6.11]. By corollary 4.13, $H \cap Z$ is a nef subvariety in Y . \square

Proposition 7.9. *Let X be a projective variety of dimension n over k . Then*

$$\overline{\text{WMov}}_1(X) = \overline{\text{Mov}}_1(X)$$

Proof. Let $\pi : Y \rightarrow X$ be a proper, surjective map, $Z \subset Y$ be a nef subvariety of dimension 1. To show that $\pi_*[Z] \in \overline{\text{Mov}}_1(X)$, it suffices to show that $D \cdot \pi_*[Z] = \pi^*D \cdot [Z] \geq 0$ for any pseudoeffective divisor on X , since the dual cone of $\overline{\text{Mov}}_1(X)$ is the cone of pseudoeffective divisors [BDPP13]. This follows from theorem 6.1. \square

Let us recall Hartshorne's conjecture A:

Conjecture 7.10 ([Har70, Conjecture 4.4]). *Let X be a smooth projective variety, and let Y be a smooth subvariety with ample normal bundle. Then $n[Y]$ moves in a large algebraic family for $n \gg 0$.*

This was disproved by Fulton and Lazarsfeld. They constructed an ample rank 2 vector bundle on \mathbb{P}^2 , such that any multiple of the zero section in the total space of the vector bundle does not move.

In view of proposition 7.6, theorem 7.8 and proposition 7.9, we make the following conjecture.

Conjecture 7.11 (Modified Hartshorne's conjecture A). *Let X be a projective variety of dimension n . Then*

$$\overline{\text{WMov}}_d(X) = \overline{\text{Mov}}_d(X)$$

for $1 \leq d \leq n - 1$.

If the conjecture is true, the cycle class of any nef subvariety of X will lie in the movable cone. The key point in the conjecture is that we only consider the cycle classes up to numerical equivalence; the movable cone is also defined to be the closure of the cone generated by movable cycles. This seems to be one of the weakest possible ways of stating the conjecture that relates positivity of the normal bundle of subvarieties and their movability.

One might want to study the closure of the convex cone generated by the cycle class of nef subvarieties of dimension d (in $N_d(X)$) instead. We now give an example where it is not of full dimension, when $d = \dim X - 1$.

Lemma 7.12 ([Ott12, Corollary 3.4]). *Let X be a normal projective variety over k . Let $Y \subset X$ be a nef subscheme of codimension 1. Then Y is a (nef) Cartier divisor.*

Proof. Let $\pi : \text{Bl}_Y X \rightarrow X$ be the blowup of X along Y , with exceptional divisor E . Then $\pi|_E : E \rightarrow Y$ is equidimensional of relative dimension 0, by 4.6. Therefore, π is quasi-finite. A proper and quasi-finite morphism is finite, so π is finite and birational, with X normal. This implies that π is in fact an isomorphism. \square

Let X be a projective variety of dimension n over k . By [Ful98, Example 19.3.3], the natural map $N^1(X) \xrightarrow{[X]} N_{n-1}(X)$ is injective. Fulger-Lehmann gave an example [FLb, Example 2.7] where $N^1(X) \xrightarrow{[X]} N_{n-1}(X)$ is not surjective. We may assume that X is normal in their example. By the above lemma, the closure of the convex cone generated by the cycle class of nef subschemes of codimension 1 lies in the subspace $N^1(X) \subsetneq N_{n-1}(X)$, hence is not full dimensional.

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